International Journal of Trend in Scientific Research and Development(IJTSRD Volume 4 Issue 4 June 2020 Available Online www.ijtsrd.com e-ISSN 2456 - 6470

Basic Maximal Total Strong Dominating Functions

Dr. M. Kavitha[1,*],
[1]Department of Mathematics, KPR Institute of Engineering and Technology, Coimbatore - 641407, India.

Let $G = (V, E)$ be a simple graph. A subset D of $V(G)$ is called a total strong dominating set of G, if for every $u \in V(G)$, there exists a $v \in D$ such that u and v are adjacent and $deg(v) \geq deg(u)$. The minimum cardinality of a total strong dominating set of G is called total strong domination number of G and is denoted by $\gamma_t^s(G)$. Corresponding to total strong dominating set of G, total strong dominating function can be defined. The minimum weight of a total strong dominating function is called the fractional total strong domination number of G and is denoted by $\gamma_{sf}^t(G)$. A study of total strong dominating functions is carried out in this paper.

Keywords: Total Strong Dominating Function, Maximal Total Strong Dominating Function, Basic Maximal Total Strong Dominating Function

Introduction: Corresponding to total strong dominating sets in a graph, total strong dominating functions may be defined. The minimality of a total strong dominating function can be characterised. Convex combination of minimal total strong dominating function is defined and studied. A total strong dominating function is basic, if it cannot be expressed as a covex combination of minimal total strong dominating functions. Basic total strong dominating functions are characterised.

Definition 0.1:
Let $G = (V, E)$ be a simple graph without strong isolates. A function $f : V(G) \to [0, 1]$ is called a **Total Strong Dominating Function (TSDF)**, if $f(N_s(u)) = \sum_{v \in N_s(u)} f(v) \geq 1, \ \forall\, u \in V(G)$, where
$N_s(u) = \{x \in N(u) \ : \ deg\ (x) \geq deg\ (u)\}$.

Definition 0.2:
A TSDF is called a **Minimal Total Strong Dominating Function (MTSDF)**, if whenever $g : V(G) \to [0, 1]$ and $g < f$, g is not a TSDF.

Definition 0.3: Let G be a graph without isolated vertex. A TSDF (MTSDF) is called a **basic TSDF** (basic MTSDF) denoted by BTSDF (BMTSDF) if it cannot be expressed as a proper convex combination of two distinct TSDF^s (MTSDF^s).

Remark 0.4: A BTSDF need not be a BMTSDF.

*Corresponding author: M. Kavitha, E-mail: kavinandhu7204@gmail.com

Lemma 0.5: Let f and g be two distinct MTSDF^s of a graph G with $B^s_f = B^s_g$ and $P_f = P_g$. Let $\delta(v) = f(v) - g(v)$, for every $v \in V$. Then

(i) If $f(v) = 0$ or $f(v) = 1$, then $\delta(v) = 0$.

(ii) $\sum_{u \in N_s(v)} \delta(v) = 0, \ \forall\ v \in B^s_f$.

(iii) $\sum_{u \in N_s(v)} \delta(v) = 0, \ \forall\ v \in B^s_g$.

Proof:

(i) We have, $f(v) = 1$ if and only if $g(v) = 1$ and

$f(v) = 0$ if and only if $g(v) = 0$.

Therefore, (i) follows.

(ii) Let $v \in B^s_f$.

Then $v \in B^s_g$.

$$\sum_{v \in N_s(v)} \delta(v) = \sum_{u \in N_s(v)} (f(u) - g(u))$$

$$= \sum_{u \in N_s(v)} f(u) - \sum_{u \in N_s(v)} g(u)$$

$$= 1 - 1 \text{ (since } v \in B^s_f \text{ and } v \in B^s_g)$$

$$= 0.$$

Therefore, (ii) follows.

(iii) is similar to (ii).

Lemma 0.6: Let f and g be convex linear combination of MTSDF^s g_1, g_2,...,g_n such that f is minimal. Then $B^s_f = B^s_g = \bigcap_{i=1}^{n} B^s_{g_i}$, $P_f = P_g = \bigcup_{i=1}^{n} P_{g_i}$ and g is minimal.

Proof:

Let $v \in P_f$.

Then $f(v) > 0$.

Let $f = \sum_{i=1}^{n} \lambda_i g_i$, $0 < \lambda_i < 1$, $\sum_{i=1}^{n} \lambda_i = 1$.

Suppose $g_i(v) = 0$, $\forall\ i$. Then $f(v) = 0$, a contradiction.

Therefore, $g_i(v) > 0$, for at least one i. Therefore, $v \in \bigcup_{i=1}^{n} P_{g_i}$.

Therefore, $P_f \subseteq \bigcup_{i=1}^{n} P_{g_i}$. Suppose $v \in \bigcup_{i=1}^{n} P_{g_i}$.

Then $g(v_i) > 0$, for some i. Therefore, $f(v) > 0$.

Therefore, $v \in P_f$. Therefore, $\bigcup_{i=1}^{n} P_{g_i} \subseteq P_f$.

Therefore, $P_f = \bigcup_{i=1}^{n} P_{g_i}$. Similarly, $P_g = \bigcup_{i=1}^{n} P_{g_i}$.

Therefore, $P_f = P_g = \bigcup_{i=1}^{n} P_{g_i}$. Let $v \in B^s_f$.

Then $f(N_s(v)) = 1$. Therefore, $\sum_{i=1}^{n} \lambda_i g_i(N_s(v)) = 1$.

Suppose $v \notin B^s_{g_i}$, for some i, $1 \leq i \leq n$. Then $g_i(N_s(v)) > 1$.

Since $g_i(N_s(v)) \geq 1$, for all j, $1 \leq j \leq n$,

$\sum_{i=1}^{n} \lambda_i g_i(N_s(v)) > \sum_{i=1}^{n} \lambda_i.$
$= 1$, a contradiction.
Therefore, $v \in B^s_{g_i}$, $\forall\ i$. Therefore, $v \in \bigcap_{i=1}^{n} B^s_{g_i}$.
Therefore, $B^s_f \subseteq \bigcap_{i=1}^{n} B^s_{g_i}$.
Let $v \in \bigcap_{i=1}^{n} B^s_{g_i}$. Then $g_i(N_s(v)) = 1$, for all i, $1 \leq i \leq n$.
Therefore, $\sum_{i=1}^{n} \lambda_i g_i(N_s(v)) = \sum_{i=1}^{n} \lambda_i = 1$.
Therefore, $f(N_s(v)) = 1$. Therefore, $v \in B^s_f$.
Therefore, $\bigcap_{i=1}^{n} B^s_{g_i} \subseteq B^s_f$. Therefore, $B^s_f = \bigcap_{i=1}^{n} B^s_{g_i}$.
Similarly, $B^s_g = \bigcap_{i=1}^{n} B^s_{g_i}$. Hence $B^s_f = B^s_g = \bigcap_{i=1}^{n} B^s_{g_i}$.
Since f is minimal, B^s_f weakly dominates P_f.
That is, $\bigcap_{i=1}^{n} B^s_{g_i}$ weakly dominates $\bigcup_{i=1}^{n} P_{g_i}$.
That is, B^s_g weakly dominates P_g.
Hence g is a MTSDF.

Theorem 0.7: Let f be a MTSDF. Then f is a BMTSDF if and only if there does not exist an MTSDF g such that $B^s_f = B^s_g$ and $P_f = P_g$.

Proof:

Suppose f is a BMTSDF.
Suppose there exists a MTSDF g such that $B^s_f = B^s_g$ and $P_f = P_g$.
Let $S = \{a \in \Re\ :\ h_a = (1 + a)g - af\}$ be a TSDF and $B^s_{h_a} = B^s_f$ and $P_{h_a} = P_f$.
Then S is a bounded open interval.
Let $S = (k_1, k_2)$.
Then $k_1 < -1 < 0 < k_2$.
Also h_{k_1} and h_{k_2} are MTSDF^s.
$h_{k_1} = (1 + k_1)g - k_1 f$.
Therefore, $f = \frac{(1 + k_1)g}{k_1} - \frac{hk_1}{k_1}$.
Let $\lambda_1 = \frac{1 + k_1}{k_1}$ and $\lambda_2 = \frac{-1}{k_1}$.
Therefore, λ_1 and λ_2 are positive and $\lambda_1 + \lambda_2 = 1$.
Therefore, f is a convex combination of g and h_{k_1}.
Therefore, f is not a BMTSDF, a contradiction.
Therefore, there does not exist a MTSDF g such that $B^s_f = B^s_g$ and $P_f = P_g$.
Conversely, suppose f is not a BMTSDF.
Then there exists MTSDF^s g_1, g_2,...,g_n such that $f = \sum_{i=1}^{n} \lambda_i g_i$,
where $0 < \lambda_i < 1$ and $\sum_{i=1}^{n} \lambda_i = 1$.

Let $g = \sum_{i=1}^{n} \mu_i g_i$, $0 < \mu_i < 1$ and $\sum_{i=1}^{n} \mu_i = 1$.

Then by lemma 0.6, $B_f^s = B_g^s = \bigcap_{i=1}^{n} B_{g_i}^s$ and $P_f = P_g = \bigcup_{i=1}^{n} P_{g_i}$ and since f is a MTSDF, g is a MTSDF.

Thus there exists a MTSDF g such that $B_f^s = B_g^s$ and $P_f = P_g$.

Hence the theorem.

Theorem 0.8: Let f be a MTSDF of a graph $G = (V, E)$ with $B_f^s = \{v_1, v_2, ..., v_m\}$ and $P_f' = \{u \in V : 0 < f(u) < 1\} = \{u_1, u_2, ..., u_n\}$.

Let $A = [a_{ij}]$ be a $m \times n$ matrix defined by

$$a_{ij} = \begin{cases} 1, & if\ v_i\ weakly\ dominates\ u_j \\ 0, & \text{otherwise.} \end{cases}$$

Consider the system of linear equations given by $\sum_{j=1}^{n} a_{ij} x_j = 0$, $1 \leq i \leq m$.

Then f is a BMTSDF if and only if the above system does not have a non-trivial solution.

Proof:

Suppose f is not a BMTSDF.

Then there exists a MTSDF g such that $B_f^s = B_g^s$ and $P_f = P_g$.

Let $x_j = f(u_j) - g(u_j)$, $1 \leq j \leq n$.

Suppose $x_j = 0$, $\forall\ j$, $1 \leq j \leq n$.

If $f(v) = 0$, then $v \notin P_f = P_g$.

Therefore, $g(v) = 0$.

Therefore, $f(v) - g(v) = 0$, $\forall\ v \notin P_f$.

That is, $f(v) = g(v)$, $\forall\ v \notin P_f$.

If $f(v) = 1$, then by Theorem **??**, $g(v) = 1$ and hence $f(v) - g(v) = 0$.

Therefore, $f = g$, a contradiction.

Therefore, there exists, some j, $1 \leq j \leq n$ such that $x_j \neq 0$.

Let $f(u_{j_1}) - g(u_{j_1}) \neq 0$.

$$\sum_{j=1}^{n} a_{ij} x_j = \sum_{j=1}^{n} a_{ij}(f(u_j) - g(u_j))$$
$$= \sum_{u \in N_s(v_i)} (f(u) - g(u))$$

(since $a_{ij} = 1$ because v_i weakly dominates u)

$= 0$, by lemma0.5.

Since $x_j \neq 0$, the left hand side has a non-trivial solution.

Conversely, let $\{x_1, x_2, ..., x_n\}$ be a non-trivial solution for the system of
linear equations.

Define $g : V(G) \to [0, 1]$ as follows:

$$g(v) = \begin{cases} f(v), & if\ v \notin P_f' \\ f(v) + \frac{x_j}{M}, & if\ v = u_j,\ 1 \leq j \leq n,\ where\ M\ is\ to\ be\ suitably\ chosen. \end{cases}$$

Since $\{x_1, x_2, ..., x_n\}$ is a non-trivial solution, $g \neq f$.

Since $0 < f(u_j) < 1$, choose $M_j > 0$ such that $0 < (f(u_j) + \frac{x_j}{M_j}) < 1$, for each j, $1 \leq j \leq n$.

Let $M' = max\{M_1, M_2, ..., M_n\}$.

Choose M to be equal to M'.

For any $v \in V$, $g(N_s(v)) = \sum_{u \in N_s(v)} g(u)$

$$= \sum_{u \in N_s(v) \cap P'_f} g(u) + \sum_{u \in N_s(v) - P'_f} g(u)$$

$$= \sum_{u_i \in N_s(v) \cap P'_f} (f(v_i) + \frac{x_i}{M'}) + \sum_{u \in N_s(v) - P'_f} f(u)$$

$$= \sum_{u \in N_s(u)} f(u) + \frac{1}{M'} \sum_i x_i$$

$$= f(N_s(v)) + \tfrac{1}{M'} \sum_i x_i$$

If $v \in B^s_f$, then $\sum_i x_i = \sum_{j=1}^{n} a_{ij} x_i = 0$.

(since v weakly dominates P_f and hence $a_{ij} = 1$).

Therefore, $g(N_s(v)) = f(N_s(v)) = 1$.

Suppose $v \notin B^s_f$.

Then $f(N_s(v)) > 1$.

Choose $M'' > 0$ such that $g(N_s(v)) > 1$, $\forall\ v \notin B^s_f$.

Let $M = max\{M', M''\}$.

For this choice of M, we have

$0 \leq g(v) \leq 1$ and $\sum_{u \in N_s(v)} g(v) \geq 1$, $\forall\ v \in V$

Therefore, g is a TSDF.

From what we have seen above,

$B^s_f = B^s_g$ and $P_f = P_g$.

Since f is a MTSDF, B^s_f weakly dominates P_f.

Therefore, B^s_g weakly dominates P_g. Therefore, g is a MTSDF.

Hence f is not a BMTSDF.

Corollary 0.9: Let $G = (V, E)$ be a graph without isolated vertices. Let S be a minimal total strong dominating set of G. Then χ_s is a BMTSDF.

Proof:

Clearly, χ_s is a MTSDF.

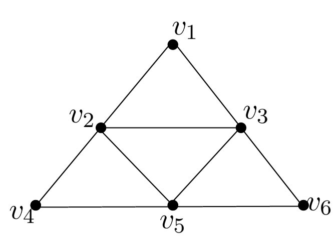
Let $f = \chi_s$.

$P'_f = \phi$.

Therefore from the above theorem, χ_s is a BMTSDF.

Example 0.10:

Consider Hajo's Graph H_3 :

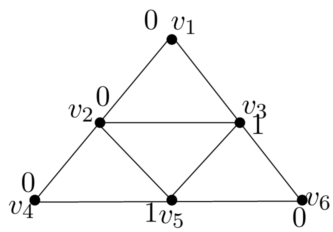


Define f_1 and f_2 as follows:

$f_1(v_1) = f_1(v_4) = f_1(v_2) = f_1(v_6) = 0$.

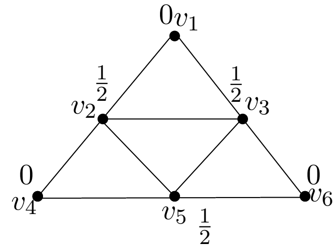
$f_1(v_3) = f_1(v_5) = 1$.

f_1 :



$f_2(v_1) = f_2(v_4) = f_2(v_6) = 0.$
$f_2(v_2) = f_2(v_3) = f_2(v_5) = \frac{1}{2}.$

f_2 :



f_1 is a TSDF.
$B^s_{f_1} = \{v_1, v_3, v_4, v_5\}.$
$P_{f_1} = \{v_3, v_5\}.$
$B^s_{f_1}$ weakly dominates P_{f_1}.
Therefore, f_1 is a MTSDF.
Therefore, f_2 is a TSDF.
$B^s_{f_2} = V.$
$B^s_{f_2}$ weakly dominates P_{f_2}.
f_2 is a MTSDF.
$P'_{f_1} = \phi.$
Let $A = [a_{ij}]_{m\times n}$ be a $m \times n$ matrix defined by

$$a_{ij} = \begin{cases} 1, & if\ v_i \in B^s_f\ weakly\ dominates\ u_j\ in\ P'_f \\ \\ 0, & otherwise. \end{cases} .$$

Consider the system of linear equations given by
$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0$
.
.
.
$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{1n}x_n = 0.$
Since $P'_{f_1} = \phi$, the system of equations does not occur and hence does not have a non-trivial solution.
Therefore, f_1 is a BMTSDF.
f_2 is a TSDF.
$B^s_{f_2} = V,\ P_{f_2} = \{v_2, v_3, v_4\}.$
$P'_{f_2} = \{v_2, v_3, v_4\}.$

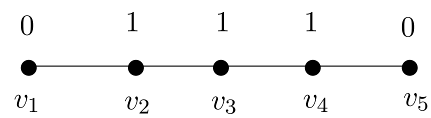
$$A = [a_{ij}]_{6\times 3} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{6\times 3}$$

$$\begin{cases} x_1 + x_2 = 0 \\ x_2 = 0 \\ x_1 = 0 \\ x_2 = 0 \end{cases} \quad imply \quad x_1 = 0 \ , \ x_2 = 0$$

Therefore, f_2 is a BMTSDF.

Example 0.11:

P_5 :



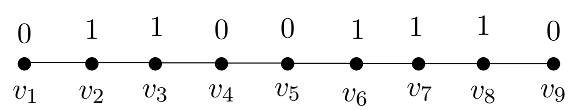
f_1 is a TSDF.
$B^s_{f_1} = \{v_1, v_2, v_4, v_5\}$.
$P_{f_1} = \{v_2, v_3, v_4\}$.
$B^s_{f_1}$ weakly dominates P_{f_1}.

$$A = [a_{ij}]_{4\times 3} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccc} v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$x_1 = 0$, $x_2 = 0$, $x_3 = 0$.
Therefore, f_1 is a BMTSDF.

Example 0.12:

Consider P_9 :



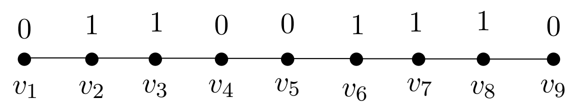
Define f_1 on $V(P_6)$ as follows:
$f_1(v_1) = f_1(v_4) = f_1(v_5) = f_1(v_9) = 0$.
$f_1(v_2) = f_1(v_3) = f_1(v_6) = f_1(v_7) = f_1(v_8) = 1$.
f_1 is a TSDF.
$B^s_{f_1} = \{v_1, v_2, v_3, v_4, v_5, v_6, v_8, v_9\}$.
$P_{f_1} = \{v_2, v_3, v_6, v_7, v_8\}$.
$B^s_{f_1}$ weakly dominates P_{f_1}.
Therefore, f_1 is a MTSDF.

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_8 \\ v_9 \end{array} \begin{array}{c} \begin{array}{ccccc} v_2 & v_3 & v_6 & v_7 & v_8 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_i = 0$, $\forall\ i$, $1 \leq i \leq 5$.
Therefore, f_1 is a BMTSDF.

Example 0.13:

Consider P_9 :



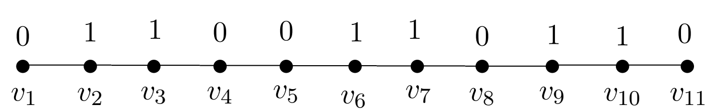
Define f_1 on $V(P_9)$ as follows:
$f_1(v_1) = f_1(v_4) = f_1(v_5) = f_1(v_9) = 0$.
$f_1(v_2) = f_1(v_3) = f_1(v_6) = f_1(v_7) = f_1(v_8) = 1$.
f_1 is a TSDF.
$B^s_{f_1} = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$.
$P_{f_1} = \{v_2, v_3, v_6, v_7, v_8\}$.
$B^s_{f_1}$ weakly dominates P_{f_1}.
Therefore, f_1 is a MTSDF.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_i = 0$, $\forall\ i$, $1 \leq i \leq 5$.
Therefore, f_1 is a BMTSDF.

Example 0.14:

Let P_{11} :



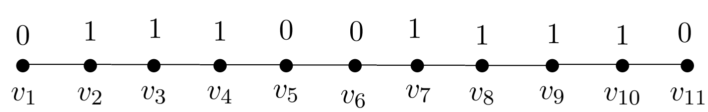
Define f_1 on $V(P_{11})$ as follows:
$f_1(v_1) = f_1(v_4) = f_1(v_5) = f_1(v_8) = f_1(v_{11}) = 0$.

$f_1(v_2) = f_1(v_3) = f_1(v_6) = f_1(v_7) = f_1(v_9) = f_1(v_{10}) = 1.$
f_1 is a TSDF.
$B^s_{f_1} = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_9, v_{10}, v_{11}\}.$
$P_{f_1} = \{v_2, v_3, v_6, v_7, v_9, v_{10}\}.$
$B^s_{f_1}$ weakly dominates P_{f_1}.
Therefore, f_1 is a MTSDF.

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_8 \\ v_9 \end{array} \begin{array}{c} \begin{array}{ccccc} v_2 & v_3 & v_6 & v_7 & v_8 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_i = 0, \forall\ i, 1 \leq i \leq 6.$
Therefore, f_1 is a BMTSDF.
Again consider

P_{11} :



Define f_2 on $V(P_{11})$ as follows:
$f_2(v_1) = f_2(v_5) = f_2(v_6) = f_2(v_{11}) = 0.$
$f_2(v_2) = f_2(v_3) = f_2(v_4) = f_2(v_7) = f_2(v_8) = f_2(v_9) = f_2(v_{10}) = 1.$
f_2 is a TSDF.
$B^s_{f_2} = \{v_1, v_2, v_4, v_5, v_6, v_7, v_{10}, v_{11}\}.$
$P_{f_2} = \{v_2, v_3, v_4, v_7, v_8, v_9, v_{10}\}.$
$B^s_{f_2}$ weakly dominates P_{f_2}.

$$B = \begin{array}{c} \\ v_1 \\ v_2 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_{10} \\ v_{11} \end{array} \begin{array}{c} \begin{array}{ccccccc} v_2 & v_3 & v_4 & v_7 & v_8 & v_9 & v_{10} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_i = 0, \forall\ i,\ 1 \leq i \leq 7$.
Therefore, f_2 is a BMTSDF.

Example 0.15:
Let $G = C_{2n+1}$.
Let $V(G) = \{v_1, v_2, ..., v_{2n+1}\}$.

Case (i)

Let $n \equiv 0\ (mod\ 2)$.
Let $n = 2k$.
Then $2n + 1 = 4k + 1$.
Let $f(v_i) = \begin{cases} 1, & if\ i \equiv 1, 2\ (mod\ 4) \\ 0, & if\ i \equiv 3, 0\ (mod\ 4). \end{cases}$
Then f is a TSDF.
$B_f^s = \{v_2, v_3, v_4, v_5, ..., v_{4k+1}\}$
$P_f = \{v_1, v_5, ..., v_{4k+1}, v_2, v_6, ..., v_{4k-2}\}$
Clearly, B_f^s weakly dominates P_f.
Therefore, f is a MTSDF.

$$A = \begin{array}{c} \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ . \\ . \\ . \\ v_{4k+1} \end{array} \begin{array}{c} \begin{array}{cccccccc} v_1 & v_2 & v_5 & v_6 & \cdots & v_{4k-1} & v_{4k-2} & v_{4k+1} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ . & & & & & & & \\ . & & & & & & & \\ . & & & & & & & \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ . \\ . \\ . \\ x_{2k+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}$$

Therefore, $x_i = 0, \forall\ i,\ 1 \leq i \leq 2k+1$.

Therefore, f is a BTMSDF.

Case (ii)

Let $n \equiv 1\ (mod\ 2)$.

Let $n = 2k+1$.

Then $2n+1 = 4k+3$.

Let $f(v_i) = \begin{cases} 1, & if\ i \equiv 1,2\ (mod\ 4) \\ 0, & if\ i \equiv 3,0\ (mod\ 4). \end{cases}$

Then f is a TSDF.

$B^s_f = \{v_1, v_2, v_5, v_6, ..., v_{4k+2}\}$.

$P_f = \{v_1, v_2, v_5, v_6, ..., v_{4k+1}, v_{4k+2}\}$.

Clearly, B^s_f weakly dominates P_f.

Therefore, f is a MTSDF.

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_5 \\ v_6 \\ . \\ . \\ . \\ v_{4k+2} \end{array} \begin{array}{ccccccc} v_1 & v_2 & v_5 & v_6 & \cdots & v_{4k+1} & v_{4k+2} \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ . & & & & & & \\ . & & & & & & \\ . & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ . \\ . \\ . \\ x_{2k+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}$$

Therefore, $x_i = 0, \forall\ i,\ 1 \leq i \leq 2k+2$.

Therefore, f is a BTMSDF.

Case (iii)

Let $G = C_{2n}$.

Let $V(G) = \{v_1, v_2, ..., v_{2n}\}$.

Let $f(v_i) = \begin{cases} 1, & if\ i \equiv 1,2\ (mod\ 4) \\ 0, & otherwise. \end{cases}$

Clearly, f is a TSDF.

$B^s_f = \begin{cases} V, & if\ n \equiv 2\ (mod\ 4) \\ V - \{v_1, v_{2n}\}, & if\ n \equiv 1\ (mod\ 4). \end{cases}$

$P_f = \begin{cases} \{v_1, v_5, ..., v_{2n-3}, v_2, v_6, ..., v_{2n-2}\}, & if\ n \equiv 2\ (mod\ 4) \\ \{v_1, v_5, ..., v_{2n-1}, v_2, v_6, ..., v_{2n}\}, & if\ n \equiv 1\ (mod\ 4). \end{cases}$

Clearly, B_f^s weakly dominates P_f.
Therefore, f is a MTSDF.
Let $n \equiv 2 \ (mod\ 4)$.

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ . \\ . \\ . \\ v_{2n} \end{array} \begin{array}{c} \begin{array}{ccccccc} v_1 & v_2 & v_5 & v_6 & \cdots & v_{2n-3} & v_{2n-2} \end{array} \\ \left[\begin{array}{ccccccc} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ . & & & & & & \\ . & & & & & & \\ . & & & & & & \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{array} \right] \end{array}$$

$AX = 0 \ implies \ X = 0.$
Therefore, f is a BMTSDF.
Let $n \equiv 1 \ (mod\ 4)$.

$$A = \begin{array}{c} \\ v_2 \\ v_3 \\ . \\ . \\ . \\ v_{2n-1} \end{array} \begin{array}{c} \begin{array}{ccccccc} v_1 & v_2 & v_5 & v_6 & \cdots & v_{2n-1} & v_{2n} \end{array} \\ \left[\begin{array}{ccccccc} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ . & & & & & & \\ . & & & & & & \\ . & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{array} \right] \end{array}$$

$AX = 0 \ implies \ X = 0.$
Therefore, f is a BMTSDF.

REFERENCES

[1] Acharya, B.D., *The Strong Domination Number of a Graph and Related Concepts*, J.Math. Phys. Sci., 14,471-475(1980).
[2] Allan R.B., and Laskar.R.C., *On Domination and Independent Domination Number of a Graph*, Dicrete Math., 23, 73-76(1978).
[3] Arumugam.S. and Regikumar.K., *Basic Minimal Dominating Functions*, Utilitas Mathematica, 77, pp. 235-247(2008).
[4] Arumugam.S. and Regikumar.K., *Fractional Independence and Fractional Domination Chain in Graphs*, AKCE J.Graphs Combin., 4, No.2, pp.161-169(2007).
[5] Arumugam.S. and Sithara Jerry.A., *A Note On Independent Domination in Graphs*, Bulletin of the Allahabad Mathematical Society, Volume 23, Part 1.(2008).
[6] Arumugam.S. and Sithara Jerry.A., *Fractional Edge Domination in Graphs*, Appl. Anal. Discrete Math.,(2009).

[7] Berge.C., *Theory of Graphs and its Applications*, Dunod, Paris(1958).
[8] Berge.C., *Graphs and Hypergraphs*, North-Holland, Amsterdam(1973).
[9] Cockayne E.J., Dawes R.M. and Hedetniemi.S.T., *Total domination in graphs*, Networks, 10, 211-219(1980).
[10] Cockayne E.J., Favaron O., Payan C. and Thomasaon A.G., *Contributions to the theory of domination, independence and irredundance in graphs*, Discrete Math., 33, 249-258(1981).
[11] Cockayne E.J., MacGillivray G. and Mynhardt C.M., *Convexity of minimal dominating functions and universal in graphs*, Bull. Inst. Combin. Appl., 5, 37-38(1992).
[12] Cockayne E.J. and Mynhardt C.M., *Convexity of minimal dominating functions of trees: A survey*, Quaestiones Math., 16, 301-317(1993).
[13] Cockayne E.J., MacGillivray G. and Mynhardt C.M., *Convexity of minimal dominating functions of trees-II*, Discrete Math., 125, 137-146(1994).
[14] Cockayne E.J. and Mynhardt C.M., *A characterization of universal minimal total dominating functions in trees*, Discrete Math., 141, 75-84(1995).
[15] Cockayne.E.J., Fricke.G., Hedetniemi.S.T. and Mynhardt.C.M., *Properties of Minimal Dominating Functions of Graphs*, Ars Combin., 41,107-115(1995).
[16] Cockayne E.J., MacGillivray G. and Mynhardt C.M., *Convexity of minimal dominating functions of trees*, Utilitas Mathematica, 48, 129-144(1995).
[17] Cockayne.E.J. and Hedetniemi.S.T., *Towards a Theory of Domination in Graphs*, Networks, 7,247-261(1977).
[18] Cockayne.E.J. and Mynhardt.C.M., *Minimality and Convexity of dominating and related functions in Graphs*, A unifying theory, Utilitas Mathematica, 51,145-163(1997).
[19] Cockayne.E.J. and Mynhardt.C.M. and Yu.B., *Universal Minimal Total Dominating Functions in Graphs*, Networks, 24, 83-90(1994).
[20] Grinstead D. and Slater P.J., *Fractional Domination and Fractional Packing in Graphs*, Congr., Numer., 71, 153-172(1990).
[21] Harary.F., *Graph Theory*, Addison-Wesley, Reading, Mass(1969).
[22] Hedetniemi.S.T. and Laskar.R.L., *Bibliography on Domination in Graphs and Some Basic Definitions of Domination Parameters*, Discrete Math., 86,pp 257-277(1990).
[23] Ore.O, *Theory of Graphs*, AMS(1962).
[24] Regikumar K., *Dominating Functions* - Ph.D thesis, Manonmaniam Sundaranar University(2004).
[25] Sampathkumar.E and Pushpa Latha.L., *Strong Weak Domination and Domination Balance in a Graph*, Discrete Math., 161,235-242(1996).
[26] Scheinerman E.R. and Ullman D.H., *Fractional Graph Theory: A rational approach to the theory of graphs*, John Wiley and Sons Inc.,(1997).
[27] Terasa W. Haynes, Stephen T. Hedetneimi, Peter J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker Inc.,(1998).
[28] Terasa W. Haynes, Stephen T. Hedetneimi, Peter J. Slater, *Domination in Graphs: Advanced Topics* Marcel Dekker Inc., New York(1998).
[29] Yu.B., *Convexity of Minimal Total Dominating Functions in Graphs*, J.Graph Theory, 24,313-321(1997).