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Circular Coloring Signed Graphs Has No Contains K_k-minor

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ABSTRACT

In 1943, Hugo Hadwiger showed that any graph that contains no K_4-minors is 3colorable. He considers any graph which has no K_{k+1}-minors is k-colorable. Based on Naserasr, Wang and Zhu's definitions of the circular chromatic number for a signed graph, particular generalized versions of Hadwiger's conjecture that might be valid in a class of sign graphs are formalized. We prove in this paper that, if the signed graph $G_{-\sigma}$ has no $(K_{k+1},-)$-minor, it means that $\chi_c(G_{-\sigma}) \leq 3$.

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1. INTRODUCTION

In the literature, Hadwiger's conjecture has been considered as one of the more interesting conjectures in graph theory. It always tends to expand the four color theorem. The conjecture asserts that all graphs without K_{k+1}-minor are k-colorables. Thus, the $3 \geq k$ case in this conjecture is quite straightforward. However, the situation where $k = 4$ implies the four-color theorem. Therefore, if $k + 1$ implies k, then the problems with the conjecture increase by only k. For this reason, in 1979, Catlin introduced a strong variant on the $k = 3$ case that we reformulate, making use of notions of a signed graph and a circular coloring. We therefore say that a signed graph (G,σ) has a minor (H,π) if an isomorphic graph (H,π) can be obtained from a subgraph of (G,σ) by the following steps: deleting vertices or edges, contracting a positive edge and switching.

Theorem 1.1 *[2] Assume that a signed graph G_- does not have $(K_4,-)$-minor, then $\chi_c(G_+) \leq$*

3.

B. Gerard, P. Seymour et al [6] have strengthened Catlin's result. The Catlin theorem 1.1 gives a generalization of the result of B. Gerard and P. Seymour [6] below: *Suppose a graph G does not contain any K_4-minors, then G is 3-colorable*. This outcome, called conjecture, is more solid than the famous Hadwiger conjecture. We call that the Odd-Hadwiger conjecture. Its use in this work is reformulated as follows.

Conjecture 1.2 *Assume that a signed graph G_- does not have $(K_{k+1},-)$-minor, then $\chi_c(G_+) \leq k$*

Hadwiger [5] and Dirac [3] have demonstrated individually that, if $k = 4$, the Conjecture 1.2 is correct. In the case of $k = 5$, Wagner [11] proved that this conjecture refconj1 is similar to the Four- color theorem. Haken and Appel proved it [1] in 1977 with computer assistance. In the case $k = 6$, Robertson et al [9] as a proof of the Hadwiger's conjecture. They also reduced it to the four color theorem.

In view of all of the above, for the sake of generalization, we can ask the following problem:

Problem 1.3 *Assume that a signed graph G_- does not have $(K_{k+1},-)$-minor, which value is k such that $\chi_c(G,-\sigma) \leq k$?*

In this paper, an answer to this question is considerably simpler and will necessarily be a direct extension of the original proof by Catlin [2], B. Gerard, P. Seymour et al [6].

2. Notion of signed graphs and its minors

A signed graph (G,σ) is defined as a graph G equipped with a signature $\sigma: E(G) \rightarrow \{+1,-1\}$ through which every edge is either negative (assigned sign -) or positive (otherwise, assigned sign+). These are binary models where we characterized the edges as either *attractive* (or even) or *repulsive* (or odd). In the following, we will write $(G,\sigma) = G_\sigma$

A subgraph $(H,\sigma_{E(H)})$ of G_σ is immediately considered to be a signed graph. $(H,\sigma_{E(H)})$ is a subgraph of the graph of G_σ. A sign applied to a particular e edge in $(H,\sigma_{E(H)})$ is the same as that applied to that e in G_σ. Finally, the subgraph G has no restrictions. It may contain multiple loops and edges, as well as half and loose edges with, respectively, one or more ends (these edges are not reported). For example, consider that X is a cycle, the result of the product of the signs on the edges is the sign of X. A subgraph $(H,\sigma_{E(H)})$ of G_σ is a signed graph resulting from removing vertices and edges.

A G_σ-signed graph allows a $(H,\sigma_{E(H)})$ if a signed graph isomorphic to $(H,\sigma_{E(H)})$ which can be derived from a subgraph of G_σ by contracting the edge identifies its endpoints in a new vertex. (G,σ) will have a minor K_k if there is a succession of vertex deletions and edge contractions resulting from K_k.

The subgraphs H and H' of disjoint vertices of the signed graph G_σ are said to be adjacent if there exists an edge of G_σ with one extremity inside $V(H)$ and the other inside $V(H')$. Otherwise, both H and H' are not considered.

3. Circular coloring Signed graphs and Minors

In the recent literature, many authors study the concept of coloring of signed graphs. Zaslavsky, in 1980s, studied vertex coloring of signed graphs (see [15]). He defines the coloring of a signed graph G_σ as a function $g: V(G) \rightarrow \{\pm k,...,0\}$ so that, for every edge $e = xy$ in G, $g(x) \neq \sigma_e g(y)$.

The idea and the way to color such a signed graph are quite simple. First of all, use signed colors in such a way that the vertices can be changed. Second, consider it so that the normal rule of coloring the adjacent vertices in various colors will be followed so long the connecting edge will be positive.

Notice that the coloring that contains $2k+1$ labels drawn from the whole set $\{-k,...,0,...,k\}$ is called a k coloring of signed graphs. On the other hand, the one containing the labels $2k$ of the set $\{-k,...,0,...,k\}$ is called a k-coloring without zero. Considering a signed graph G, according to Zaslavsky in [15], a proper vertex coloring of G, or just a *coloration* is such an application $f: V(G) \rightarrow Z$ that for each edge $e = uv$ of G the color $\sigma(u)$ is distinct from the color $\sigma(e)f(v)$, where $\sigma(e)$ denotes the sign of e.

Called star chromatic number, in 1988 Vince introduced a circular chromatic number $\chi_c(G)$ of a graph G, see [10].

He generalized this concept naturally to the chromatic number of a graph. The notion of "circular chromatic number" was several studied in [14] and the preceding definition has been given in [12].

A circular chromatic number $\chi_c(G)$ of a signed graph G_σ is the smallest ratio $r = \frac{p}{q}$ for which one (p,q)-coloring of (G,σ) exists.

In the literature, one of the most important results of the circular chromatic number is: for any graph G, $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$ and thus $\chi(G) = \lceil\chi_c(G)\rceil$. Notice that Xuding Zhu, in 2001 and 2006, studied this circular chromatic number of graphs, see [12, 13].

In 2018, authors Kang and Steffen came up with the idea of introducing the idea of the circular coloring of the signed graphs [7].

To have "antipodal" points is a different definition of the concept from the one we would use in this paper. Both definitions use points in a circle as colors. The discrete version in [7] is using Z_k as the colors.

The elements of Z_k can be considered as uniformly spread out points on the circle). In [7], a diameter of a steady state circle is chosen. The antipode to a point will be obtained by rotating the circle around the designated diameter. The colors are not symmetrical in such a coloring.

Indeed, having each two extremities of the diameter chosen, his antipodal value is itself. This definition in the article [7] extends further the signed graph coloring which admits 0. Zaslavsky being the one who brought the notion of opposition to the coloring without 0. This notion states that 0 is a special color, whose antipode is itself 0. In what follows, we consider specialty in a particular color as an unwanted characteristic. An element which is circular should be rotation invariant. In this respect, the circular graph coloring described in this article extends the circular coloring of signed graphs more precisely.

Consider $(H,\sigma_{E(H)})$ a signed graph where $V(H) = \{V_i\}$ with $i \in [k]$. A model $(H,\sigma_{E(H)})$ in a signed graph G_σ is a collection of connected subgraphs with disjoint vertices H_i with i $in[k]$ such that for $i \neq j \in \{1,2,...,k\}$ where $V_iV_j \in E(G)$, there is an edge of which one ends in H_i and the other ends in H_j. Reversing the contraction, we can see that G_σ has a $(H,\sigma_{E(H)})$-minor if and only if there is a model of $(H,\sigma_{E(H)})$ within G_σ. Therefore G_σ has a K_k if and only if there is a substructure (i.e. a model of K_k).

Notation 3.1

1. K_k is a complete graph with k vertices.
2. G_+ is a signed graph in which all edges have positive values.
3. G_- is a signed graph in which all edges have negative values.
4. $K_k \leq G$ to mean G contains K_k.
5. $K_k \leq_m G$ to signify G having a K_k-minor.
6. $K_k \leq_t G$ to mean G contains a subdivision of K_k.
7. $\delta_G(X)$ means that all edges have precisely one extremity in X.

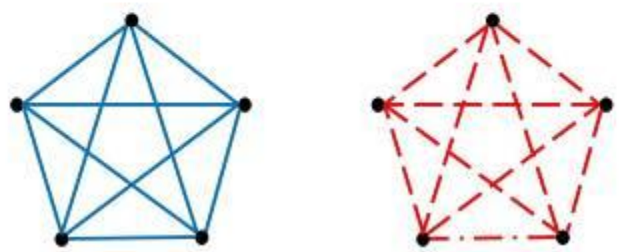

Figure 1: Examples of signed graphs: K_5^+ blue line and K_5^- dote red line

The main focus of our paper is on identifying K_k-minors in the signed graphs. The following lemma gives a more concrete definition of a K_k-minor; the result is well known so we will skip the simple proof.

Lemma 3.2 *Consider the signed graph G_σ. G_σ has a K_k-minor if, and only if, by writing $V(K_k) = \{v_1,...,v_k\}$, there exists disjoint nonempty subsets $V_1,...,V_k$ of $V(G)$ such that $G[V_i]$ is connected for all i with $e(V_i,V_j) > 0$ at each time $v_iv_j \in E(K_k)$.*

Lemma 3.3 *Consider the signed graph G_σ. G_σ has K_k-minor if, and only if, there are disjoint signed trees with vertices $(T_1,...,T_k)$ in G_σ and some set $V(G) \supseteq X$ such that: 1. for every $i \in \{1,...,n\}$ $\delta_G(X) \supseteq E(T_i)$;*

2. $\forall (1 \leq i < j \leq k)$, $\exists (uv) \in E(G) - \delta_G(X)$ where $u \in V(T_i)$, $v \in V(T_j)$.

Theorem 3.4 *Assume that a signed graph G_- does not have $(K_{k+1},-)$-minor, then $\chi_c(G_{-\sigma}) \leq 3$*

4. Main theorem and some results

In this section, we first prove two useful lemmas 4.1 and 4.2 and theorem 4.3. We then conclude the section with the proof of Theorem 3.4.

A vertex at the top of a graph is one adjacent to all the others. When v was a vertex on graph G and $G-v$ has K_k-minor, obviously G also has K_{k+1}-minor. Similar results for signed graphs are less obvious.

Lemma 4.1 *Consider G_σ is a signed graph, v a vertex of G_σ, and be $H = G-v$. When (H,σ) has K_k-minor, G_σ has K_{k+1}-minor.*

Proof. By the lemma 3.3, there exist disjoint signed trees at vertices $(T_1,...,T_k)$ in G_σ, and a set $X \subseteq V(G)$ such that :

1. for each $i \in \{1,...,n\}$ $\delta_G(X) \supseteq E(T_i)$;
2. $\forall (1 \leq i < j \leq k)$, $\exists (uv) \in E(G) - \delta_G(X)$ with $u \in V(T_i)$ and $v \in V(T_j)$.

Let us consider i,j distinct. By (2), we cannot have $X \supseteq V(T_i)$ and $V(T_j) - X = \emptyset$. Therefore, replacing X by $V(H)-X$, we can assume $V(T_i)-X = \emptyset$, $\forall i \in 1,...,k$. Let T_{k+1} be the signed tree in G_σ consisting of the single vertex v. Now through the lemma 3.3, G_σ has a K_{k+1}-minor.

Lemma 4.2 *Consider G_σ and (K_k,π) as signed graphs, where $V(K_k) = \{v_1,...,v_k\}$. Let (K_k,π) be a minor of G_σ, and have this be proved by the disjoint nonempty subsets $V_1,...,V_k$ from Lemma' 3.2. Assume that no proper sub-graph of G_σ contains (K_k,π) as a minor. Then*

1. Each $G[V_i]$ is minimally connected, i.e., a signed tree
2. Whenever $v_iv_j \in E(K_k)$, we have $e(V_i,V_j) = 1$.
3. Whenever $v_iv_j \notin E(K_k)$ then $e(V_i,V_j) = 0$.
4. For every leaf w of a signed tree $G[V_i]$ of size larger than 1, we can find some $j \neq i$ such that $e(\{w\},V_j) > 0$ 5. $G[V_i]$ has at most $d(v_i)$ leaves
5. $V_1,...,V_k$ cover V (G).

Proof. If the first assertion were false, then we would be able to delete an edge of V_i to get a proper sub-graph $G^1{}_\sigma$ of G_σ, where $G^1[V_i]$ is still connected, thus the same sub-sets $V_1,...,V_k$ would testify that the proper subgraph $G^1{}_\sigma$ has a K_k-minor.

If the second assertion were false, so there would be at least two edges from some V_i to some V_j, and if we remove one of them, we get a proper subgraph $G^2{}_\sigma$ of G_σ, where $G[V_i] = G^2V_i]$ for each i, and there is still an edge in $G^2{}_\sigma$ from V_i to V_j every time $v_iv_j \in E(K_k)$. So the same subsets $V_1,...,V_k$ would testify that the proper sub-graph $G^2{}_\sigma$ has a K_k-minor.

If the third assertion were false, then we might find $v_iv_j \notin E(H)$ with an edge between V_i and V_j. Deleting this edge gives a suitable sub-graph $G^3{}_\sigma$ of G_σ, where $G[V_i] = G^3[V_i]$ for each i. So the same subsets $V_1,...,V_k$ would testify that the proper sub-graph $G^2{}_\sigma$ has a K_k-minor.

If the fourth statement were false, then we can delete w to obtain a signed tree $G[V_i]-w = G[V_i \setminus w]$. This is always non-empty and connected, and there is always an edge in the proper subgraph $G^4 = G - w$ of G_σ from $V_i \setminus \{w\}$ to V_j whenever $v_i v_j \in E(K_k)$. Thus, the subsets $V\ 1,...,V_i\ \setminus\{w\},...,V_k$ would show that the proper subgraph $G^4 = G - w$ has a K_k-minor.

We shall use the claims above to prove the fifth assertion. With no loss $G[V_i]$ has more than one vertex (otherwise $G[V_i]$ has a single vertex, of degree zero it is not a leaf). There exists an injective function of the set of $G[V_i] \rightarrow v_j$, where v_j is a neighbor of v_i. This set is yielded by sending a leaf w at any fixed choice of j as given by the fourth claim. Since $e(\{w\},V_j) > 0$, by the third claim v_j is a neighbor of v_i, so this function is well defined. It is an injection since if any two leaves w,w' are sent to the same j, then there are edges of w in V_j and w' in V_j. By the second claim, $w = w'$.

If the sixth claim were false, then $V_1,...,V_k$ would testify that the proper subgraph $G[V_1 \cup ... \cup V_k]$ of G_σ has a K_k-minor.

Now we are prepared to resolve the theorem 1.1; for convenience, we rephrase it here to the contrapositive.

Theorem 4.3 *For every signed graph G_-, and when G_- is non-circular 3-colorable, G_- has a $(K_{k+1},-)$ -minor.*

Proof. This result is true when $k = 2$. We suppose the result holds for $k = n - 1 \geq 2$, and consider the case when $k = n$. Consider $(G,-)$ as a 3-colorable non-circular signed graph. We do not lose any generality by considering $(G,-)$ as connected. So consider $v \in V(G)$, and consider T as a signed tree with width $(G,-)$ expanded by v. Now, for every $i \in$ N let $V_i \subseteq V(G)$ denote the collection of vertices at distance i from v in T and let (Hi,π) represent the subgraph of $(G,-)$ that is induced from V_i.

Consider $C^* = E(G) - (E(H_1) - E(H_2))$. Since C^* is a cut, restriction of G_- to C^* is circular 2-colorable. Then since G_- is not circular 3-colorable, $G - C^*$ is not circular 3colorable. Then the components of $G - C^*$ are $(H_0,H_1,...)$, so there is $i \in$ N so that H_i is non circular 3-colorable. By induction hypothesis (Hi,π) has a K_k-minor. Suppose G'_- was obtained by adding a vertex to (Hi,π). Notice that G'_- is a minor of G_σ. By the lemma 4.1, G'_- has K_{k+1}-minor, as does G_σ.

Proof of Theorem 3.4

Suppose G_- is a signed graph with no $(K_{k+1},-)$-minor such that $\chi_c(G_{-\sigma}) \geq 4$. Then G_- contains some 4-contraction complete signed graph as a minor. Without loss of generality, we may assume that G_- is chosen to be 4-contraction complete signed graph. By Theorem 4.3, $(G,-)$ as a 3-colorable non-circular signed graph. We do not lose any generality by considering $(G,-)$ as connected.So consider $v \in V(G)$, and consider T as a signed tree with width $(G,-)$ expanded by v. Now, for every $i \in$N let $V_i \subseteq V(G)$ denote the collection of vertices at distance i from v in T and let (Hi,π) represent the subgraph of $(G,-)$ that is induced from V_i. From lemma 4.1, G'_- has K_{k+1}-minor, as does G_σ. By the lemma 3.3, there exist disjoint signed trees at vertices $(T_1,...,T_k)$ in G_σ, and a set $X \subseteq V(G)$ such that :

1. for each $i \in \{1,...,n\}$ $\delta_G(X) \supseteq E(T_i)$;
2. $\forall (1 \leq i < j \leq k), \exists (uv) \in E(G) - \delta_G(X)$ with $u \in V(T_i)$ and $v \in V(T_j)$.

Let us consider i,j distinct. By (2), we cannot have $X \supseteq V(T_i)$ and $V(T_j) - X = \emptyset$. Therefore, replacing X by $V(H) - X$, we can assume $V(T_i) - X = \emptyset, \forall i \in 1,...,k$. Let T_{k+1} be the signed tree in G_σ consisting of the single vertex v. Now through the lemma 3.3, G_σ has a K_{k+1}-minor. Consider $C^* = E(G) - (E(H_1) - E(H_2))$. Since C^* is a cut, restriction of G_- to C^* is circular 2-colorable. Then since G_- is not circular 3-colorable, $G - C^*$ is not circular 3-colorable. Then the components of $G - C^*$ are $(H_0,H_1,...)$, so there is $i \in$ N so that H_i is non circular 3-colorable. By induction hypothesis (Hi,π) has a K_k-minor. In particular, there must be two vertices of degree 4 which are not adjacent, and so G_- contains two different K_5 subgraphs. Since G_- is 3-connected by Theorem 4.3, it follows from lemma 4.1 G_- has K_{k+1}-minor, as does G_σ, a contradiction. This contradiction completes the proof.

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