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A Note on Weakly β-Continuous Functions in Tritopological Spaces

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ABSTRACT

As a generalization of β-continuous functions and weakly β-continuous functions in bitopological space, we introduce and study some properties of weakly β-continuous functions in tritopological spaces and we obtain its some characterizations.

KEYWORDS: *Tritopological spaces, Continuous functions, Triowise regular, Triowise closure, and Triowise connected sets*

Mathematics Subject Classification: 54D10, 54D15, 54A05, 54C08.

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1. INTRODUCTION

This is the third in a series of our M. Phil papers. The first and the second such papers have appeared in 2015 [22] and in 2016 [23]. In general topology the notation of semi-preopen sets due to Andrijevic [14] or β-open sets due to Mashhour et al.[13] plays a significant role. In [13] the concept of β-continuous functions is introduced and further Popa and Noiri [9] studied the concept of weakly β-continuous functions. In 1992, Khedr et al. [8] introduced and studied β-continuity in bitopological spaces. In 2008, Sanjay Tahiliani [15] introduced and studiedweakly β-continuous functions in bitopological spaces. In this paper, we introduce and study the notation of weakly β-continuous functions in tritopological spaces and investigate several properties of these functions in tritopological spaces.

We have used the definitions and terminology of text book of S. Majumdar and N. Akhter [16], Munkres [17], Dugundji [18], Simmons [19], Kelley [20] and Hocking-Young [21].

2. Basic Definitions

In the present paper, the space (X, P_1, P_2, P_3), (X, σ_1, σ_2) and (X,T) denote the tritopological, bitopological and topological spaces respectively.

Let (X, T) be a topological space and A be a subset of X. The closure and interior of A are denoted by Cl(A) and Int(A) respectively.

In (X, σ_1, σ_2) the closure and interior of $A \subseteq X$ with respect to σ_i are denoted by iCl(A) and iInt(A) respectively, for i=1, 2.

In (X, P_1, P_2, P_3) the closure and interior of $A \subseteq X$ with respect to $P_i$are denoted by iCl(A) and iInt(A) respectively, for i=1,2,3.

Definition 2.1: A subset A of a tritopological space (X, P_1, P_2, P_3) is said to be

1. (i,j,k)-regular open([1]) if A=iInt(jCl(kInt(A))), where $i \neq j \neq k$, i,j,k=1,2,3.

2. (i,j,k)-regular closed([3]) if A=iCl(jInt(kCl(A))), where $i\neq j\neq k$, i,j,k=1,2,3.
3. (i,j,k)-semi-open([2]) if A⊂iCl(jInt(kCl(A))), where $i\neq j\neq k$, i,j,k=1,2,3.
4. (i,j,k)-preopen([5]) if A⊂iInt(jCl(kInt(A))),where $i\neq j\neq k$, i,j,k=1,2,3.

Definition 2.2: A subset A of a tritopological space (X, P_1, P_2, P_3) is said to be (i,j,k)-semi-preopen ([8]) if there exists a (i,j,k)-preopen set U such that U⊆A⊆jCl(kInt(U)) or it is said to be (i,j,k)-β-open if A⊆jCl(kInt(iCl(A))), where $i\neq j\neq k$, i,j,k=1,2,3.

The complement of (i,j,k)-semi-preopen set is said to be (i,j,k)-semi-preclosed [8] or is said to be (i,j,k)-β-closed if iInt(jCl(kInt(A)))⊆A,where $i\neq j\neq k$, i,j,k=1,2,3.

Lemma 2.1: Let (X, P_1, P_2, P_3) be a tritopological space and $\{A_\lambda : \lambda\in\Delta\}$ be a family of subsets of X. Then

1. if A_λ is (i,j,k)-β-open for each $\lambda\in\Delta$, then $\bigcup_{\lambda\in\Delta} A_\lambda$ is (i,j,k)-β-open

2. if A_λ is (i,j,k)-β-closed for each $\lambda\in\Delta$, then $\bigcap_{\lambda\in\Delta} A_\lambda$ is (i,j,k)-β-closed.

Proof: (1) The proof follows from Theorem 3.2of [8].

(2)This is an immediate consequence of (1).

Definition 2.3: Let A be a subset of a tritopological space (X, P_1, P_2, P_3)

1. The (i,j,k)-β-closure [8] of A, denoted by (i,j,k)-βCl(A) is defined to be the intersection of all (i,j,k)-β-closed sets containing A.
2. The (i,j,k)-β-interior of A, denoted by (i,j,k)-βInt(A) is defined to be the union of all (i,j,k)-β-open sets contained in A.

Lemma 2.2: Let (X, P_1, P_2, P_3) be a tritopological space and A be a subset of X. Then

1. (i,j,k)-βInt(A) is (i,j,k)-β-open
2. (i,j,k)-βCl(A) is (i,j,k)-β-closed
3. A is (i,j,k)-β-open iff A=(i,j,k)-βInt(A)
4. A is (i,j,k)-β-closed iff A=(i,j,k)-βCl(A).

Proof: (1) and (2) are obvious from Lemma 2.1, (3) and (4) are obvious from (1) and (2).

Lemma 2.3: For any subset A of a tritopological space (X, P_1, P_2, P_3), $x\in$ (i,j,k)-βCl(A) if and only if $U\cap A\neq\phi$ for every (i,j,k)-β-open set U containing x.

Proof: The proof is trivial.

Lemma 2.4: Let (X, P_1, P_2, P_3) be a tritopological space and A be a subset of X. Then

1. X-(i,j,k)-βInt(A)=(i,j,k)-βCl(X-A)
2. X-(i,j,k)-βCl(A)=(i,j,k)-βInt(X-A).

Proof: (1) By Lemma 2.2, (i,j,k)-βCl(A) is (i,j,k)-β-closed. Then X-(i,j,k)-βCl(A) is (i,j,k)-β-open. On the other hand, X - (i,j,k)-βCl(X-A)⊆A and hence X - (i,j,k)-βCl(X-A)⊆(i,j,k)-βInt(A). Conversely, let $x\in$ (i,j,k)-βInt(A). Then there exists (i,j,k)-β-open set G such that $x\in$ G⊆A. Then X-G is (i,j,k)-β-closed and X-A⊆X-G. Since $x\notin$ X-G, $x\notin$ (i,j,k)-βCl(X-A) and hence (i,j,k)-βInt(A)⊆X-(i,j,k)-βCl(X-A).

Hence X-(i,j,k)-βInt(A)=(i,j,k)-βCl(X-A).

(2)This follows immediately from (1).

Definition 2.4: Let (X, P_1, P_2, P_3) be a tritopological space and A be a subset of X. A point x of X is said to be in the (i,j,k)-θ-closure [6] of A, denoted by (i,j,k)-cl_θ(A) if for every i-open set U containing x, A∩jCl(kInt(U))≠ϕ, where $i\neq j\neq k$, i,j,k=1,2,3.

A subset A of X is said to be (i,j,k)-θ-closed if A=(i,j,k)-cl_θ(A). A subset A of X is said to be (i,j,k)-θ-open if X-A is (i,j,k)-θ-closed.

The (i,j,k)-θ-interior of A, denoted by (i,j,k)-Int_θ(A)is defined as the union of all (i,j,k)-θ-open sets contained in A. Hence $x\in$ (i,j,k)-Int_θ(A) if and only if there exists a i-open set U containing x such that $x\in$ U⊆jCl(kInt(U))⊆A.

Lemma 2.5: For any subset A of a tritopological space (X, P_1, P_2, P_3) the following properties hold:

1. X-(i,j,k)-Int_θ(A) =(i,j,k)-cl_θ(X-A)

2. X-(i,j,k)-cl_θ(A)=(i,j,k)-Int_θ(X-A).

Lemma 2.6: [6] Let (X, P_1, P_2, P_3) be a tritopological space. If U is a k-open set of X, then (i,j,k)- cl_θ(U) = iCl(jInt(U)).

Definition 2.5: A mapping f : (X,P_1, P_2, P_3) → (Y,Q_1, Q_2, Q_3) is said to be (i,j,k)-β-continuous[8] if $f^{-1}(V)$ is (i,j,k)-β-open in X for each Q_i-open set V of Y.

Example 2.1: Consider the following tritopologies on X = $\{a,b,c\}$ and Y= $\{p,q,r\}$ respectively:

$P_1 = \{X, \phi, \{a\}, \{a,b\}\}$, $P_2 = \{X, \phi, \{a\}, \{b,c\}\}$, $P_3 = \{X, \phi, \{a\}, \{b\} \{a,b\}\}$

and $Q_1 = \{Y, \phi, \{p\}, \{p,r\}\}$, $Q_2 = \{Y, \phi, \{p\}\}$, $Q_3 = \{Y, \phi, \{q\}\}$

We define the mapping f : (X,P_1, P_2, P_3) → (Y,Q_1, Q_2, Q_3) by f(a) = p, f(b) = q and f(c) = r. Then f is (1,2,3)-β-continuous since the inverse of each member of the topology Q_i - on Y is a (1,2,3)-β-open set in (X,P_1, P_2, P_3).

Definition 2.6: (1) A mapping f : (X,P_1, P_2, P_3) → (Y,Q_1, Q_2, Q_3) is said to be (i,j,k)- weakly precontinuous if for each $x \in$ X and each Q_i-open set V of Y containing f(x), there exists (i,j,k)- preopen set U containing x such that f(U) ⊆ jCl(kInt(V)).

Example 2.2: Consider the following tritopologies on X = $\{a,b,c\}$ and Y = $\{p,q,r\}$ respectively:

$P_1 = \{X, \phi, \{a\}, \{a,b\}\}$, $P_2 = \{X, \phi, \{a\}, \{b,c\}\}$, $P_3 = \{X, \phi, \{a\}, \{b\} \{a,b\}\}$

and $Q_1 = \{Y, \phi, \{p\}, \{p,r\}\}$, $Q_2 = \{Y, \phi, \{p\}\}$, $Q_3 = \{Y, \phi, \{q\}\}$. We define the mapping f : (X, P_1, P_2, P_3) → (Y, Q_1, Q_2, Q_3) by f(a) = p, f(b) = q and f(c) = r. Then f is (1,2,3)- weakly precontinuous. Since if $a \in$ X and Q_1-open set V= $\{p,r\}$, then we have (1,2,3)- preopen set U = $\{a\}$ such that f(U) ⊆ Q_2-Cl(Q_3-Int(V)).

(2)A mapping f : (X, P_1, P_2, P_3) → (Y, Q_1, Q_2, Q_3) is said to be (i,j,k)- weakly- β- continuous if for each $x \in$ X and each Q_i -open set V of Y containing f(x), there exists (i,j,k)- β-open set U containing x such that f(U) ⊆ jCl(kInt(V)).

Remark 2.1: Since every (i,j,k)-preopen set is (i,j,k)-β-open ([8] Remark 3.1), every (i,j,k)- weakly precontinuous function is (i,j,k)-weakly- β-continuous for i ≠ j ≠ k, i,j,k=1,2,3. The converse is not true.

3. Characterization

Theorem 3.1: For a mapping f : (X, P_1, P_2, P_3) → (Y, Q_1, Q_2, Q_3) the following properties are equivalent:

1. f is (i,j,k)- weakly- β-continuous.
2. (i,j,k)- β-Cl(f^{-1}(kInt(jCl(iInt(B))))) ⊆ f^{-1}(jCl(iInt(B))) for every subset B of Y
3. (i,j,k)- β-Cl(f^{-1}(kInt(F))) ⊆ f^{-1}(F) for every (i,j,k)- regular closed set F of Y
4. (i,j,k)- β-Cl(f^{-1}(Cl(V))) ⊆ f^{-1}(iCl(jInt(B))) for every Q_k-open set V of Y
5. f^{-1}(V) ⊆ (i,j,k)- β-Int(f^{-1}(jCl(kInt(V)))) for every Q_i-open set V of Y

Proof: (1) ⟹ (2). Let B be any subset of Y. Suppose that $x \in$ X- f^{-1}(jCl(iInt(B))). Then $f(x) \in$ Y-jCl(iInt(B)) so that there exists a Q_j- open set V of Y containing f(x) such that V ∩ B = ϕ, so V ∩ kInt(jCl(iInt(B))) = ϕ and hence kCl(iInt(V)) ∩ kInt(jCl(iInt(B))) = ϕ. Therefore ∃ (i,j,k)- β- open set U containing x such that f(U) ⊆ kCl(iInt(V)). Hence we have U ∩ f^{-1}(kInt(jCl(iInt(B)))) = ϕ and $x \in$ X- (i,j,k)- β-Cl(f^{-1}(kInt(jCl(iInt(B))))) by Lemma 2.6. Thus we have (i,j,k)- β-Cl(f^{-1}(kInt(jCl(iInt(B))))) ⊆ f^{-1}(jCl(iInt(B))).

(2) ⟹ (3) Let F be any (i,j,k)-regular closed set of Y. Then F = iCl(jInt(kCl(F))) and we have (i,j,k)- β-Cl(f^{-1}(kInt(F)))=(i,j,k)- β-Cl(f^{-1}(kInt(iCl(jInt(kCl(F)))))) ⊆ iCl(jInt(kCl(F)))) = f^{-1}(F).

(3) ⟹ (4) For any Q_k- open set V of Y, iCl(jInt(V)) is (i,j,k)-regular closed. Then we have (i,j,k)- β-Cl(f^{-1}(Cl(V))) ⊆ (i,j,k)- β-Cl(f^{-1}(kInt(iCl(V)))) ⊆ f^{-1}(iCl(jInt(V))).

(4)$\Rightarrow$(5) Let V be Q_i- open set of Y. Then Y - jCl(kInt(V)) is σ_k - open set in Y and we have (i,j,k)- β-Cl(f^{-1}(Y-jCl(kInt(V)))) $\subseteq f^{-1}$(jCl(Y-jCl(kInt(V))) and hence X- (i,j,k)- β-Int(f^{-1}(jCl(kInt(V))))$\subseteq$ X- f^{-1}(jInt(kCl(V)))$\subseteq$X- f^{-1}(V). Thus we obtain f^{-1}(V) $\subseteq$(i,j,k)- β-Int(f^{-1}(jCl(kInt(V)))).

(5)$\Rightarrow$(1) Let $x \in$ X and V be a Q_i- open set containing f(x). We have $x \in f^{-1}$(V) $\subseteq$(i,j,k)- β-Int(f^{-1}(jCl(kInt(V)))). Put U= (i,j,k)- β-Int(f^{-1}(jCl(kInt(V)))). By Lemma 2.5, U is (i,j,k)- β- open set containing x and f(U)$\subseteq$jCl(kInt(V)). This shows that f is (i,j,k)- weakly- β-continuous.

Theorem 3.2: For a mapping f : (X, P_1, P_2, P_3) → (Y, Q_1, Q_2, Q_3) the following properties are equivalent:

1. f is (i,j,k)- weakly- β-continuous
2. f((i,j,k)- βCl(A)) $\subseteq$ (i,j,k)- $Cl_\theta(f(A))$ for every subset A of X
3. (i,j,k)- βCl(f^{-1}(B)) $\subseteq f^{-1}$((i,j,k)- $Cl_\theta(f(B))$) for every subset B of Y
4. (i,j,k)- βCl(f^{-1}(jInt((i,j,k)- $Cl_\theta(f(B))$))) $\subseteq f^{-1}$((i,j,k)- $Cl_\theta(f(B))$) for every subset B of Y.

Proof: (1)$\Rightarrow$ (2). Suppose that f is (i,j,k)- weakly- β-continuous. Let A be any subset of X, $x \in$ (i,j,k)- βCl(A) and V be a Q_i- open set of Y containing f(x). Then there exists (i,j,k)- β- open set U containing x such that f(U)$\subseteq$jCl(kInt(V)). Since $x \in$ (i,j,k)- βCl(A), by Lemma 2.3, we obtain U$\cap$ A $\neq \phi$ and hence $\phi \neq$ f(U) $\cap$ f(A)$\subseteq$jCl(kInt(V)) $\cap$ f(A). Therefore, we obtain f(x)$\in$ (i,j,k)- $Cl_\theta(f(A))$.

(2)$\Rightarrow$ (3). Let B be any subset of Y. Then we have f((i,j,k)- βCl(f^{-1}(B))) $\subseteq$ (i,j,k)- $Cl_\theta\left(f^{-1}(B)\right)$)$\subseteq$(i,j,k)- $Cl_\theta(B)$and hence (i,j,k)- βCl(f^{-1}(B)) $\subseteq f^{-1}$((i,j,k)- $Cl_\theta(f(B))$).

(3)$\Rightarrow$ (4). Let B be any subset of Y. Since (i,j,k)- $Cl_\theta(B)$ is Q_i- closed set in Y, by Lemma 2.6, (i,j,k)- βCl(f^{-1}(jInt((i,j,k)- $Cl_\theta(B)$))) $\subseteq f^{-1}$((i,j,k)- Cl_θ(jInt((i,j,k)- $Cl_\theta(B)$)) = f^{-1}(iCl(jInt((i,j,k)- Cl_θ(B)))) $\subseteq f^{-1}$(iCl(i,j,k)- Cl_θ(B)) = f^{-1}((i,j,k)- Cl_θ(B)).

(4)$\Rightarrow$ (1). Let V be any Q_k- open set of Y. Then by Lemma 2.10, V $\subseteq$jInt(iCl(V))=jInt((i,j,k)- $Cl_\theta(V)$) and we have (i,j,k)- βCl(f^{-1}(V)) $\subseteq$(i,j,k)- βCl(f^{-1}(jInt((i,j,k)- Cl_θ(B))) $\subseteq f^{-1}$((i,j,k)- Cl_θ(V))= f^{-1}(iCl(jInt(V))). Thus we have (i,j,k)- βCl(f^{-1}(V)) $\subseteq f^{-1}$(iCl(jInt(V))). It follows from Theorem 3.1 that f is (i,j,k)- weakly-β-continuous.

Definition 3.1: A tritopological space (X, P_1, P_2, P_3) is said to be (i,j,k)-regular ([7]) if for each $x \in$ X and each P_i- open set U containing x, $\exists$ a P_i- open set V such that $x \in$ V $\subseteq$jCl(kInt(V)) $\subseteq$U.

Example 3.1: Consider the following tritopologies on X$=\{$a,b,c,d$\}$:

$P_1=\{X,\phi,\{a\},\{a,b,c\},\{a,c\},\{a,d\}\}$, $P_2=\{X,\phi,\{a,b\},\{b,c\},\{b\}\}$, $P_3=\{X,\phi,\{a,c\},\{b,c\},\{c\}\}$

Then X is (1,2,3) regular since for $a \in$ X, P_1- open set U = $\{a,b,c\}$, there exists P_1- open set V = $\{a,c\}$ such that V $\subseteq P_2$ –Cl(P_3- Int(V)) $\subseteq$U.

Lemma 3.1 : [10] If a tritopological space (X, P_1, P_2, P_3) is (i,j,k)-regular, then (i,j,k)- Cl_θ(F) = F for every P_i- closed set F.

Theorem 3.3: Let (Y, Q_1, Q_2, Q_3) be an (i,j,k)-regular tritopological space. For a mapping f : (X, P_1, P_2, P_3) → (Y, Q_1, Q_2, Q_3) the following properties are equivalent:

1. f is (i,j,k)- β-continuous
2. f^{-1}((i,j,k)- Cl_θ(B)) is (i,j,k)- β- closed in X for every subset B of Y
3. f is (i,j,k)- weakly - β-continuous

4. f^{-1}(F) is (i,j,k)- β- closed in X for every (i,j,k)- θ- closed set F of Y
5. f^{-1}(V) is (i,j,k)- β- open in X for every (i,j,k)- θ- open set V of Y.

Proof : (1) $\Rightarrow$ (2). Let B be any subset of Y. Since (i,j,k)- $Cl_\theta(B)$ is Q_i- closed set in Y, it follows by Theorem 5.1 of [8] that f^{-1}((i,j,k)- Cl_θ(B)) is (i,j,k)- β- closed in X.

(2) $\Rightarrow$ (3). Let B be any subset of Y. Then we have (i,j,k)- βCl(f^{-1}(B))) $\subseteq$ (i,j,k)- βCl(f^{-1}(i,j,k)- Cl_θ(B)) = f^{-1} (i,j,k)- $Cl_\theta(B)$. By Theorem 3.2, f is (i,j,k)- weakly - β-continuous.

(3) $\Rightarrow$ (4). Let F be any (i,j,k)- θ- closed set of Y. Then by Theorem 3.2, (i,j,k)- βCl(f^{-1}(F)) $\subseteq f^{-1}$ ((i,j,k)- Cl_θ(F)) = f^{-1}(F). Therefore by Lemma 2.5, f^{-1}(F) is (i,j,k)- β- closed in X.

(4) $\Rightarrow$ (5). Let V be any (i,j,k)- θ- open set of Y. By (4), $f^{-1}(Y - V) = X - f^{-1}$(V) is (i,j,k)- β- closed in X and hence f^{-1}(V) is (i,j,k)- β- open in X.

(5) $\Rightarrow$ (1). Since Y is (i,j,k)- regular, by Lemma 3.4, (i,j,k)- Cl_θ(B) = B for every Q_i- closed set B of Y and hence Q_i- open set is (i,j,k)- θ- open set. Therefore f^{-1}(V) is (i,j,k)- β- open for every Q_i- open set V of Y. By Theorem 5.1 of [8], f is (i,j,k)- β-continuous.

4. Weakly - β-continuity and β-continuity

Definition 4.1: A mapping f : (X, P_1, P_2, P_3) $\rightarrow$ (Y, Q_1, Q_2, Q_3) is said to be (i,j,k)- weakly* quasi continuous (briefly w*.q. c)[10] if for every Q_i- open set V of Y, f^{-1}(jCl(kInt(V))) is triclosed in X.

Theorem 4.1: If a mapping f : (X, P_1, P_2, P_3) $\rightarrow$ (Y, Q_1, Q_2, Q_3) is (i,j,k)- weakly- β-continuous and (i,j,k)- w*.q. c, then f is (i,j,k)- β-continuous.

Proof : Let $x \in$ X and V be any Q_i- open set of Y containing f(x). Since f is (i,j,k)- weakly- β- continuous, there exists an (i,j,k)- β- open set U of X containing x such that f(U) $\subseteq$ jCl(kInt(V)).

Hence $x \notin f^{-1}$(jCl(kInt(V)) - V). Therefore $x \in$ U - f^{-1}(jCl(kInt(V)) - V) = U $\cap$ (X - f^{-1}(jCl(kInt(V)) - V)). Since U is (i,j,k)- β - open and X - f^{-1}(jCl(kInt(V)) - V) is triopen, by Theorem 3.3 of [8], G = U $\cap$X - f^{-1}(jCl(kInt(V)) - V) is (i,j,k)- β - open. Then $x \in$ G and f(G) $\subseteq$ V. For if $y \in$ G, then $f(y) \notin$ (jCl(kInt(V)) - V) and hence f(y) $\in$ V. Therefore f is (i,j,k)- β-continuous.

Definition 4.2: A mapping f : (X,P_1, P_2, P_3) $\rightarrow$ (Y,Q_1, Q_2, Q_3) is said to have a (i,j,k)- β- interiority condition if (i,j,k)- βInt(f^{-1}(jCl(kInt(V))) $\subseteq f^{-1}$(V) for every Q_i- open set V of Y.

Theorem 4.2: If a mapping f : (X, P_1, P_2, P_3) $\rightarrow$ (Y, Q_1, Q_2, Q_3) is (i,j,k)- weakly- β-continuous and satisfies the (i,j,k)- β- interiority condition then f is (i,j,k)- β-continuous.

Proof: Let V be Q_i- open set of Y. Since f is (i,j,k)- weakly- β-continuous, by Theorem 3.1, f^{-1} (V) $\subseteq$ (i,j,k)- βInt(f^{-1}(jCl(kInt(V))). By (i,j,k)- β- interiority condition of f, we have (i,j,k)- βInt(f^{-1}(jCl(kInt(V))) $\subseteq f^{-1}$(V) and hence (i,j,k)- βInt(f^{-1}(jCl(kInt(V))) $= f^{-1}$(V). By Lemma 2.5, f^{-1}(V) is (i,j,k)- β- open in X and thus f is (i,j,k)- β-continuous.

Definition 4.3: Let (X, P_1, P_2, P_3) be a tritopological space and let A be a subset of X. The (i,j,k)- β- frontier of A is defined as (i,j,k)- βFr(A) = (i,j,k)- βCl(A)$\cap$(i,j,k)- βCl(X -A) = (i,j,k)- βCl(A) - (i,j,k)- βInt(A).

Theorem 4.3: The set of all points x of X for which a mapping f : (X, P_1, P_2, P_3) $\rightarrow$ (Y, Q_1, Q_2, Q_3) is not (i,j,k)- weakly- β-continuous is identical with the union of the (i,j,k)- β- frontier of the inverse images of the jCl(kInt(V)) of Q_i- open set V of Y containing f(x).

Proof : Let x be a point of X at which f(x)is not (i,j,k)- weakly- β-continuous. Then there exists a

Q_i- open set V of Y containing f(x) such that U ∩ (X - f^{-1}(jCl(kInt(V)))) $\neq \phi$ for every (i,j,k)- β- open set U of X containing x. By Lemma 2.6, $x \in$ (i,j,k)-βCl(X - f^{-1}(jCl(kInt(V)))). Since $x \in f^{-1}$(jCl(kInt(V))), we have $x \in$ (i,j,k)-βCl(f^{-1}(jCl(kInt(V)))) and hence $x \in$ (i,j,k)-βFr(f^{-1}(jCl(kInt(V)))).

Conversely, if f is (i,j,k)- weakly-β-continuous at x, then for each Q_i- open set V of Y containing f(x), there exists (i,j,k)- β- open set U containing x such that f(U) $\subseteq$jCl(kInt(V)) and hence $x \in$ U $\subseteq f^{-1}$(jCl(kInt(V))). Therefore we obtain that $x \in$ (i,j,k)- βInt(f^{-1}(jCl(kInt(V)))). This contradicts that $x \in$ (i,j,k)- βFr(f^{-1}(jCl(kInt(V)))).

5. Weakly -β-continuity and almost β-continuity

Definition 5.1: A mapping f : (X, P_1, P_2, P_3) → (Y, Q_1, Q_2, Q_3) is said to be (i,j,k)- almost β-continuous if for each $x \in$ X and each Q_i- open set V containing f(x), there exists an (i,j,k)- β- open set U of X containing x such that f(U) $\subseteq$ iInt(jCl(kInt(V))).

Lemma 5.1 : A mapping f : (X, P_1, P_2, P_3) → (Y, Q_1, Q_2, Q_3) is (i,j,k)- almost β- continuous if and only if f^{-1}(V) is (i,j,k)- β- open for each (i,j,k)-regular open set V of Y.

Definition 5.2: A tritopological space (X, P_1, P_2, P_3) is said to be (i,j,k)- almost regular [12] if for each $x \in$X and each (i,j,k)-regular open set U containing x, there exists an (i,j,k)-regular open set V of X such that $x \in$ V $\subseteq$jCl(kInt(V)) $\subseteq$U.

Theorem 5.1: Let a tritopological space (Y, Q_1, Q_2, Q_3) be (i,j,k)- almost regular. Then a mapping f : (X, P_1, P_2, P_3) → (Y, Q_1, Q_2, Q_3) is(i,j,k)- almost β- continuous if and only if it is (i,j,k)- weakly- β-continuous.

Proof: Necessity. This is obvious.

Sufficiency. Let us suppose that f is (i,j,k)- weakly-β-continuous. Let V be any (i,j,k)-regular open set of Y and $x \in f^{-1}$(V). Then we have f(x)∈ V. By the almost (i,j,k)-regularity of Y, there exists an (i,j,k)-regular open set V_0 of Y such that $f(x) \in V_0 \subseteq$jCl(kInt(V_0)) $\subseteq$V. Since f is (i,j,k)-weakly-β-continuous, there exists an (i,j,k)-β-open set U of X containing x such that f(U) $\subseteq$jCl(kInt(V_0))$\subseteq$V. This follows that $x \in$ U $\subseteq f^{-1}$(V). Therefore we have f^{-1}(V) $\subseteq$(i,j,k)-βInt(f^{-1}(V). By Lemma 2.5, f^{-1}(V) is (i,j,k)-β- open and by Lemma 5.2, f is (i,j,k)- almost β-continuous.

Definition 5.3: A tritopological space (X, P_1, P_2, P_3) is said to be triowise β-Hausdorff or triowise β-T_2 if for each distinct points x, y, z of X, there exists (i,j,k) β- open set U, V and W containing x, y and z respectively such that U∩V∩W=ϕ for i$\neq$j$\neq$k, i,j,k=1,2,3.

Theorem 5.2: Let (X, P_1, P_2, P_3) be a tritopological space. If for each distinct points x, y, z in X, there exists mapping f of (X, P_1, P_2, P_3) into triowise Hausdorff tritopological space (Y, Q_1, Q_2, Q_3) such that

1. f(x) $\neq$ f(y) $\neq$ f(z)
2. f is (i,j,k)- weakly- β-continuous at x
3. f is (j,k,i)-almost-β-continuous at y
4. f is (k,i,j)- almost- β-continuous at z.

Then (X, P_1, P_2, P_3) is triowise β-Hausdorff.

Proof: Let x, y, z be three distinct points in X. Since Y is triowise Hausdorff, there exists a Q_i- open set U containing f(x) and a Q_j- open set V containing f(y) and a Q_k- open set W containing f(z) such that U∩V∩W=ϕ. Since U, V and W disjoint we have kCl(U) ∩kInt(jCl(V)) ∩kCl(jInt(iCl(W)) = ϕ. Since f is (i,j,k)- weakly-β-continuous at x, there exists an (i,j,k) β- open set U_x of X containing x such that f(U_x)$\subseteq$kCl(U). Since f is (j,k,i)- almost-β-continuous at y, there exists an (j,k,i) β- open set U_y of X containing y such that f(U_y)$\subseteq$kInt(jCl(V)) and since f is (k,i,j)- β- open set U_z of X containing z such that f(U_z)$\subseteq$kCl(jInt(iCl(W))). Hence we have $U_x \cap U_y \cap U_z = \phi$. This shows that (X, P_1, P_2, P_3) is triowise β-Hausdorff.

6. Some Properties

Definition 6.1: A tritopological space (X, P_1, P_2, P_3) is said to be triowise Urysoshn [4] if for each distinct points x, y, z, there exists i - open set U, j-open set V and k-open set W such that $x \in U$, $y \in V$ and $z \in W$ and jCl(U)∩kCl(V)∩iCl(W) $=\phi$ for $i \neq j \neq k$, i,j,k=1,2,3.

Theorem 6.1: If (Y, Q_1, Q_2, Q_3) is triowise Urysoshn and f : (X, P_1, P_2, P_3) → (Y, Q_1, Q_2, Q_3) is triowise weakly β- continuous injection, then (X, P_1, P_2, P_3) is triowise $\beta - T_2$.

Proof: Let x, y, z be three distinct points of X. Then since f is injection, f(x)≠ f(y)≠f(z). Since Y is triowise Urysoshn, there exists P_i- open set U, P_j-open set V and P_k- open set W such that f(x) ∈ U, f(y) ∈ V and f(z) ∈ W and jCl(U)∩kCl(V)∩iCl(W) $=\phi$ for $i \neq j \neq k$, i,j,k=1,2,3. Hence f^{-1}(jCl(U)) ∩ f^{-1}(kCl(V)) ∩ f^{-1}(iCl(W)) $=\phi$. Therefore (i,j,k)- βInt(f^{-1}(jCl(U))) ∩ (j,k,i)- βInt(f^{-1}(kCl(V))) ∩ (k,i,j)- βInt(f^{-1}(iCl(W))) $=\phi$. Since f is triowise weakly β- continuous by Theorem 3.1, $x \in f^{-1}$(U) ⊆ (i,j,k)- βInt(f^{-1}(jCl(U))), $y \in f^{-1}$(V) ⊆ (j,k,i)- βInt(f^{-1}(kCl(V))), $z \in f^{-1}$(W) ⊆ (k,i,j)- βInt(f^{-1}(iCl(W))). This implies that (X, P_1, P_2, P_3) is triowise $\beta - T_2$.

Definition 6.2: A tritopological space (X, P_1, P_2, P_3) is said to be triowise connected [11] (resp. triowise β-connected) if it can not be expressed as the union of three non-empty disjoint sets U,V and W such that U is i-open, V is j-open and W is k-open (resp. (i,j,k)- β-open, (j,k,i)- β-open and (k,i,j)- β-open).

Theorem 6.2: If a mapping f : (X, P_1, P_2, P_3) → (Y, Q_1, Q_2, Q_3) is triowise weakly β- continuous surjection and (X, P_1, P_2, P_3) is triowise β-connected, then (Y, Q_1, Q_2, Q_3) is triowise connected.

Proof: Suppose that (Y, Q_1, Q_2, Q_3) is not triowise connected. Then there exists a Q_i- open set U, Q_j- open set V and Q_k- open set W such that U≠ϕ ,V ≠ϕ, W ≠ϕ, U∩V∩W=ϕ and U∪V∪W= Y. Since f is surjective , f^{-1}(U), f^{-1}(V) and f^{-1}(W) are non-empty. Moreover , f^{-1}(U) ∩ f^{-1}(V) ∩ f^{-1}(W) = ϕ and f^{-1}(U) ∪ f^{-1}(V) ∪ f^{-1}(W) = X. Since f is triowise weakly β- continuous, by Theorem 3.1, we have f^{-1}(U) ⊆ (i,j,k)- βInt(f^{-1}(jCl(U))), f^{-1}(V) ⊆ (j,k,i)- βInt(f^{-1}(kCl(V))) and f^{-1}(W) ⊆ (k,i,j)- βInt(f^{-1}(iCl(W))). Since U is j-closed and k-closed, V is i-closed and k-closed, W is i-closed and j-closed, we have f^{-1}(U) ⊆ (i,j,k)- βInt(f^{-1}(U)), f^{-1}(V) ⊆ (j,k,i)- βInt(f^{-1}(V)) and f^{-1}(W) ⊆ (k,i,j)- βInt(f^{-1}(W)). Hence f^{-1}(U) = (i,j,k)- βInt(f^{-1}(U)), f^{-1}(V) = (j,k,i)- βInt(f^{-1}(V)) and f^{-1}(W) = (k,i,j)- βInt(f^{-1}(W)). By Lemma 2.5, f^{-1}(U) is (i,j,k)- β-open, f^{-1}(V) is (j,k,i)- β-open, f^{-1}(W) is (k,i,j)- β-open in (X, P_1, P_2, P_3). This shows that (X, P_1, P_2, P_3) is not triowise β-connected.

Definition 6.3: A subset A of a tritopological space (X, P_1, P_2, P_3) is said to be (i,j,k)-quasi H-closed relative to X [1] if for each cover $\{U_\alpha : \alpha \in J\}$ of A by P_i-open sets of X, there exists a finite subset J_0 of J such that A $\subseteq \bigcup \{jCL(kInt(U_\alpha)) : \alpha \in J_0\}$.

Definition 6.4: A subset A of a tritopological space (X, P_1, P_2, P_3) is said to be (i,j,k)- β-compact relative to X if every cover of A by (i,j,k)- β-open sets of X has a finite subcover.

Theorem 6.3: If f : (X, P_1, P_2, P_3) → (Y, Q_1, Q_2, Q_3) is triowise weakly β- continuous and A is (i,j,k)- β-compact relative to X, then f(A) is (i,j,k)-quasi H-closed relative to Y.

Proof : Let A be (i,j,k)- β-compact relative to X and $\{V_\alpha : \alpha \in J\}$ any cover of f(A) by Q_i-open sets of (Y, Q_1, Q_2, Q_3). Then f(A)$\subseteq \bigcup \{V_\alpha : \alpha \in J\}$ and so A$\subseteq \bigcup \left\{ f^{-1}(V_\alpha) : \alpha \in J \right\}$. Since f is (i,j,k)-weakly β- continuous, by Theorem 3.1, we have

$f^{-1}(V_\alpha) \subseteq$ (i,j,k)-βInt(f^{-1}(jCl(kInt(V_α))) for each $\alpha \in J$. Therefore A $\subseteq \bigcup$ {(i,j,k)-βInt(f^{-1}(jCl(kInt(V_α)))} for each $\alpha \in J$. Since A is (i,j,k)-β-compact relative to X and (i,j,k)-βInt(f^{-1}(jCl(kInt(V_α))) is (i,j,k)-β-open for each $\alpha \in J$, there exists a finite subset J_0 of J such that A $\subseteq \bigcup${(i,j,k)-βInt(f^{-1}(jCl(kInt(V_α))) :$\alpha \in J_0$}. This implies that f(A)$\subseteq \bigcup${f((i,j,k)-βInt(f^{-1}(jCl(kInt(V_α)))):$\alpha \in J_0$}$\subseteq \bigcup${f(f^{-1}(jCl(kInt(V_α))): $\alpha \in J_0$ } $\subseteq \bigcup${jCl(kInt(V_α)) : $\alpha \in J_0$ }. Hence f(A) is (i,j,k)-quasi H-closed relative to Y.

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