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Quadruple Series Equations Involving Heat Polynomials

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ABSTRACT

In this paper an exact solution of the quadruple series equations involving heat polynomials Pn,υ(x, t) is given. We have also shown the solution of the quadruple series equations involving generalized Laguerre polynomials as a special case of the equations considered in the present paper. In this paper, we explore a novel class of quadruple series equations involving heat polynomials, which are central to various problems in mathematical physics, particularly in heat conduction and diffusion phenomena. Heat polynomials, known for their role in solving the heat equation, provide a robust framework for modeling and analyzing heat distribution over time. The study introduces and derives quadruple series equations that extend the classical series solutions, integrating the properties of heat polynomials to address more complex boundary conditions and multidimensional problems.

KEYWORDS: *Quadruple series equations, heat polynomials, ordinary heat polynomial, Hermite polynomial of even order, generalized Laguerre polynomial, orthogonality relation*

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1. INTRODUCTION

Heat polynomials have long been a fundamental tool in the study of heat conduction and diffusion processes, playing a crucial role in the solution of the heat equation. These polynomials, which are explicitly constructed to satisfy the heat equation, provide a powerful means of describing the evolution of temperature distributions over time [1]. The mathematical elegance and utility of heat polynomials have made them a subject of extensive research, particularly in their application to series solutions of differential equations. In recent years, there has been growing interest in extending the classical methods involving heat polynomials to more complex systems, particularly those governed by higher-dimensional and multi-variable scenarios [2-5]. One such extension is the development of quadruple series equations, which involve four independent series, each incorporating heat polynomials. These quadruple series offer a rich mathematical structure that can model more intricate physical phenomena, especially in contexts where traditional series solutions fall short [1].

In the present section, we consider the following quadruple series equations:

$$\sum_{n=o}^{\infty} \frac{A_n\, t^{-n}\ \rho^{-n}}{\Gamma\left(\upsilon+\frac{1}{2}+n+p\right)}\ P_{n+p,\upsilon}(x,-t) = f(x,t);\ o \le x < y, \qquad [1]$$

$$\sum_{n=o}^{\infty} \frac{A_n}{\Gamma\left(\mu+\frac{1}{2}+n+p\right)}\ P_{n+p,\mu}(\xi,-\rho) = \phi(x,t);\ y < x < z, \qquad [2]$$

$$\sum_{n=o}^{\infty} \frac{A_n}{\Gamma\left(\mu+\frac{1}{2}+n+p\right)}\ P_{n+p,\mu}(\xi,-\rho) = \Psi(x,t);\ z < x < h, \qquad [3]$$

$$\sum_{n=o}^{\infty} \frac{A_n}{\Gamma\left(\mu+\frac{1}{2}+n+p\right)}\ P_{n+p,\sigma}(x,-t) = g(x,t);\ h < x < \infty, \qquad [4]$$

Where, f (x,t), ϕ (x,t), Ψ (x, t) and g (x,t) are prescribed functions for $t \geq \rho > o$ and A_n is to be determined and $P_{n,\upsilon}$ (x, t) is the heat polynomials (Haimo, 1966) defined by [6-7]

$$P_{n,\upsilon}(x,t) = \sum_{k=o}^{n} 2^{2k} \binom{n}{k} \frac{\Gamma(\upsilon + \frac{1}{2} + n)}{\Gamma\left(\upsilon + \frac{1}{2} + n - k\right)} x^{2n-2k}\ t^{k}, \upsilon > o; \qquad [5]$$

It may be noted that $P_{n,0}(x, t) = \upsilon_{2n}(x,t)$ is the ordinary heat polynomial of even order defined by Rosenbloom and widder (1959) and that [8]

$P_{n,0}(x, -1) = (-1)^n.\ 2^{2n}\ (n)!\ L_n^{-1/2}\ (\frac{x^2}{4}) = H_{2n}\ (\frac{x}{2})$, the Hermite polynomial of even order defined by Erdelyi (1953).

We also define

$$W_{n,\upsilon}(x,t) = t^{-2n}\ G_\upsilon(x,t)\ P_{n,\upsilon}(x,-t) > o, \qquad [6]$$

Where, $G_\upsilon(x,t) = (2t)^{-\upsilon-1/2} \exp.(-x^2/4t)$, [7]

and $W_{n,\upsilon}(x,t)$ is the Appell transform of $P_{n,\upsilon}(x, -t)$.

The analysis is purely formal and no attempt is made to supply details of rigours proof [9-10].

2. CERTAIN INTEGRAL AND SERIES REPRESENTATIONS

The heat polynomial $P_{n,\upsilon}(x,t)$ is related to the generalized Laguerre polynomial by

$$P_{n,\upsilon}(x,-t) = (-1)^n\ 2^{2n}\ (n)!\ t^n\ L_n^{(\upsilon-\frac{1}{2})}\left(\frac{x^2}{4t}\right), \qquad [8]$$

Using the orthogonality relation for $L_n^{(\alpha)}(x)$, it can easily be verified that for t>o,

$$\int_o^\infty w_{m,\upsilon}(x,t)\ p_{n,\upsilon}\ (x,-t)\ d\,R(x) = \delta_{m,n}\ k_n\ , \qquad [9]$$

Where $d\,R(x) = 2^{1/2-\upsilon}\ [\ \Gamma(\upsilon+1/2)]^{-1}\ .x^{2\upsilon}\ dx$ [10]

$$\text{and}\quad K_n = \frac{\Gamma\left(\upsilon+\frac{1}{2}\right)}{\left[2^{\,4n}(n)!\,\Gamma\left(\upsilon+\frac{1}{2}+n\right)\right]} \qquad [11]$$

Now, using the formula (27) of [1, p.190] in the form

$$\left(\frac{d}{dx}\right)^m \left\{x^{\alpha+m}\ L_n^{(\alpha+m)}(x)\right\} = \frac{\Gamma(\alpha+m+n+1)}{\Gamma(\alpha+n+1)}\ x^{\alpha}\ L_n^{(\alpha)}(x), \qquad [12]$$

and from the relation [8], we obtain at once that

$$\left(\frac{d^2}{dx^2}\right)^m \left\{x^{2\upsilon+2m-1}\ p_{n,\upsilon+m}(x,-t)\right\} = \frac{\Gamma\left(\upsilon+\frac{1}{2}+m+n\right)}{\Gamma\left(\upsilon+\frac{1}{2}+n\right)}\ (x^{2\upsilon-1})p_{n,\upsilon}\ (x,-t) \qquad [13]$$

The relation

$$e^{-x}\ L_n^{(\alpha)}(x) = (-1)^m \left(\frac{d}{dx}\right)^m \left\{e^{-x}\ L_n^{(\alpha-m)}(x)\right\}, n \geq o, \qquad [14]$$

together with [8]yields

$$(-4t)^m \left(\frac{d^2}{dx^2}\right)^m \left\{e^{-x^2/4t}\ p_{n,\upsilon}(x,-t)\right\} = e^{-(x^2/4t)}\ p_{n,\upsilon}(x,-t) \qquad [15]$$

Now, we derive a few fractional integral type representations for $P_{n,\upsilon}(x,-t)$ and $w_{n,\upsilon}(x,t)$. Using the definition of Beta function and integrating the series for $P_{n,\upsilon}(x,-t)$ term by term with respect to x, it can easily be seen that

$$P_{n,\upsilon+\beta}(\xi,-t) = 2\xi^{-2\upsilon-2\beta+1} \frac{\Gamma\left(\beta+\upsilon+\frac{1}{2}+n\right)}{\Gamma(\beta)\Gamma\left(\upsilon+\frac{1}{2}+n\right)}$$

$$\int_o^\xi x^{2\upsilon}\left(\xi^2 - x^2\right)^{\beta-1} p_{n,\upsilon}\ (x,-t)dx\,,\ \ (\beta > 0\,, \upsilon > -\tfrac{1}{2}) \qquad [16]$$

Using the following form of the Beta function formula

$$\int_0^\infty \upsilon^{-\lambda-s}(\upsilon-\sigma)^{\lambda-\mu-1}\,d\upsilon = \frac{\Gamma(\lambda-\mu)\Gamma(\mu+s)}{\Gamma(\lambda+s)}\,\sigma^{-s-\mu} \qquad [17]$$

where $\lambda > \mu$ and $s+\mu>0$ and integrating the series for $p_{n,\upsilon}(x,-t)$ term by term with respect to t , we get

$$\int_\sigma^\infty t^{-n-\mu-1/2}(t-\sigma)^{\mu-\upsilon-1}\; p_{n,\upsilon}(x,-t)\,dt =$$

$$\frac{\Gamma(\mu-\upsilon)\Gamma\left(\upsilon+\frac{1}{2}+n\right)}{\Gamma\left(\mu+\frac{1}{2}+n\right)}\,\sigma^{-(\upsilon+\frac{1}{2}+n)}P_{n,\mu}(x,-\sigma);\ \left(\mu>\upsilon>-\frac{1}{2}\right), \qquad [18]$$

Expressing $P_{n,\upsilon}(x,-t)$ in terms of generalized Lagurerre polynomial by means of [3.1.8]and using the formula

$$\int_\xi^\infty e^{-x}(x-\xi)^{\beta-1}\,L_n^{(\alpha)}(x)dx = \Gamma(\beta)\,e^{-\xi}L_n^{(\alpha-\beta)}(\xi),\ \ (\alpha+1>\beta>0) \qquad [19]$$

It can be proved that

$$P_{n,\upsilon-\beta}(\xi,t) = 2^{1-\beta}\frac{t^{-\beta}}{\Gamma(\beta)}e^{(\xi^2/4t)}\int_\xi^\infty x\left(x^2-\xi^2\right)^{\beta-1}W_{n,\upsilon}(x,t)dx$$

$$(\upsilon+1/2>\beta>0) \qquad [20]$$

Now, we derive certain series representation for $P_{n,\upsilon}(x,t)$. Using the generating relation to Haimo (1966).

$$(1-4zt)^{-\upsilon-\frac{1}{2}}\frac{e^{x^2z}}{(1-4zt)} = \sum_{n=0}^{\infty}\frac{z^n}{(n)!}\,p_{n,\upsilon}(x,t) \qquad [21]$$

and the relation

$$(1-4zt)^{-\upsilon-\frac{1}{2}}\ \frac{e^{x^2z}}{(1-4zt)^{-(\upsilon-\mu)}}(1-4zt)^{-\frac{1}{2}-\mu}\frac{e^{x^2z}}{(1-4zt)},$$

It follows that

$$P_{n,\upsilon}(x,t) = \frac{\Gamma(n+1)}{\Gamma(\upsilon-\mu)}\sum_{k=0}^{n}\frac{\Gamma(\upsilon-\mu+n-k)\,(4t)^{n-k}}{\Gamma(n-k+1)\Gamma(k+1)}.\,p_{k,\mu}(x,t), \qquad [22]$$

Equation [22]can be inverted to get

$$2^{1/2-\mu}\,x^{2\mu}\,t^{-2k}\ \ G_\mu(x,t)\ P_{k,\mu}(x,-t) = \sum_{n=k}^{\infty}\frac{\Gamma\left(\upsilon-\mu+n-k\right)\Gamma(\mu+\frac{1}{2}+k)}{\Gamma(\upsilon-\mu)\Gamma(n-k+1)\Gamma\left(\upsilon+\frac{1}{2}+n\right)}$$

$$(-\tfrac{t}{4})^{n-k}\,2^{1/2-\upsilon}\,x^{2\upsilon}\,t^{-2n}\ \ G_\upsilon(x,t)\,P_{n,\upsilon}(x,-t),\ \ (2\mu>\upsilon) \qquad [23]$$

Equation [23] follows from [22], since they each are equivalent to

$$\int_0^\infty 2^{\frac{1}{2}-\mu}\ \ x^{2\upsilon}\,t^{-2k}\ \ G_\mu(x,t)\ P_{k,\mu}(x,-t)\ P_{n,\upsilon}(x,-t)\,dx$$

$$= \frac{\Gamma\left(\upsilon-\mu+n-k\right)\Gamma(\mu+\frac{1}{2}+k)}{\Gamma(\upsilon-\mu)\Gamma(n-k+1)}\ (-t)^{n-k}\,4^{n+k}\,(n)! \qquad [24]$$

3. SOLUTION OF QUADRUPLE SERIES EQUATIONS

Multiplying equations [1] by $(t)^{-(p+m+\mu+\frac{1}{2})}(t-\alpha)^{\mu+m-\upsilon-1}$,

Where m is a positive integer , integrating with respect to t from p to ∞ and using [18] , we get

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma\left(\mu+m+n+p+\frac{1}{2}\right)} \quad P_{n+p,\mu+m}\,(\xi,-\rho) \quad = F(\xi,\rho)\ ,0<\xi<y, \qquad [25]$$

where $\mu + m > \upsilon > -1/2$ and

$$F(\xi,\rho) = \frac{\rho^{\upsilon+\frac{1}{2}}}{\Gamma(\mu+m-\upsilon)} \int_{\rho}^{\infty} t^{-\left(p+m+\mu+\frac{1}{2}\right)} (t-\rho)^{\mu+m-\upsilon-1} f(\xi,t)dt \qquad [26]$$

Further, setting t = ρ in [4], multiplying it by $x(x^2-\xi^2)^{\sigma-\mu-1}e^{-x^2/4}$, integrating with respect to x from ξ to ∞ and applying [20] , we obtain

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma\left(\mu+n+p+\frac{1}{2}\right)} \; P_{n+p,\mu}\;(\xi-\rho) = \; G(\xi,\rho)\ ,y<\xi<\infty, \qquad [27]$$

where $s > m > -\frac{1}{2}$ *and*

$$G(\xi,\rho\,) = \; 2^{1-2(\sigma-\mu)}\,\rho^{-(\sigma-\mu)} \frac{e^{(\xi^2/4\rho)}}{\Gamma(\sigma-\mu)} \int_{\xi}^{\infty} x\left(x^2-\xi^2\right)^{\sigma-\mu-1} e^{-\left(\frac{x^2}{4\rho}\right)} \; g(x,\xi)dx \qquad [28]$$

Now, multiplying [25] by $\xi^{2(\mu+m)-1}$ *and applying the operator* $\left(\frac{d^2}{d\xi^2}\right)^m$,

We see in view of [13]that

$$\sum_{n=0}^{\infty}\frac{A_n}{\Gamma\left(\mu+\frac{1}{2}+n+p\right)} \quad P_{n+p,\mu}\,(\xi-\rho) = H\,(\xi,\rho)\,, \qquad 0<\xi<y, \qquad [29]$$

Where

$$H(\xi,\rho) \; = \xi^{1-2\mu}\left(\frac{d^2}{d\xi^2}\right)^m \left[\xi^{2(\mu+m)-1}\,F(\xi,\rho)\right] \qquad [30]$$

The left hand sides of [29] , [2] , [3] and [27] are now identical and an application of the orthogonality relation [9] yields the solution of the equations [1] , [2] , [3] and [4] in the form:

$$A_n = \frac{\Gamma\left(\mu+\frac{1}{2}\right)}{2^{4(n+p)}\,(n+p)!}\left[\int_0^y W_{n+p,\mu\,(\xi,\rho)H(\xi,\rho)dR(\xi)+} \int_y^z W_{n+p,\mu\,(\xi,\rho)\phi(\xi,\rho)dR(\xi)}\right.$$

$$\left.\int_z^h W_{n+p,\mu\,(\xi,\rho)\psi(\xi,\rho)dR(\xi)+} \int_h^{\infty} W_{n+p,\mu\,(\xi,\rho)G(\xi,\rho)dR(\xi)}\right] \qquad [31]$$

where, H(ξ,ρ) and G(ξ,ρ) are the same as defined by [30] and [28] respectively and dR(ξ) is defined by [10].

Using the relation [8] and setting [11-12]

$B_n = A_n(-1)^{n+p}\,2^{2(n+p)}(n+p)!$, we find that the equations [1] , [2] , [3] and [4] transform into

$$\sum_{n=0}^{\infty}\frac{B_n\,t^n}{\Gamma\left(\upsilon+\frac{1}{2}+n+p\right)} L_{n+p}^{\left(\upsilon-\frac{1}{2}\right)}\left(\frac{x^2}{4t}\right) = t^{-\rho}\,f(x,t), \qquad 0<x<y, \qquad [32]$$

$$\sum_{n=0}^{\infty}\frac{B_n\,t^n}{\Gamma\left(\mu+\frac{1}{2}+n+p\right)} L_{n+p}^{\left(\upsilon-\frac{1}{2}\right)}\left(\frac{x^2}{4t}\right) = t^{-\rho}\,\phi(x,t), \qquad y<x<z, \qquad [33]$$

$$\sum_{n=0}^{\infty} \frac{B_n t^n}{\Gamma\left(\mu+\frac{1}{2}+n+p\right)} L_{n+p}^{\left(\upsilon-\frac{1}{2}\right)}\left(\frac{x^2}{4t}\right) = t^{-\rho}\,\Psi(x,t), \qquad z < x < h, \qquad [34]$$

$$\sum_{n=0}^{\infty} \frac{B_n t^n}{\Gamma\left(\mu+\frac{1}{2}+n+p\right)} L_{n+p}^{\left(\upsilon-\frac{1}{2}\right)}\left(\frac{x^2}{4t}\right) = t^{-\rho}\,g(x,t), \qquad h < x < \infty, \qquad [35]$$

where , $t \geq \rho > 0$, *and their solution is given by*

$$B_n = \frac{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma(n+p+1)}{2^{\upsilon+\frac{1}{2}}\,\rho^{\upsilon+\frac{1}{2}+2n}} \left[\int_0^y e^{-(\xi^2/4\rho)} L_{n+p}^{\left(\upsilon-\frac{1}{2}\right)}\left(\frac{\xi^2}{4\rho}\right) H(\xi,\rho)\,dR(\xi) + \int_y^z e^{-(\xi^2/4\rho)} L_{n+p}^{\left(\upsilon-\frac{1}{2}\right)}\left(\frac{\xi^2}{4\rho}\right) \phi(\xi,\rho)\,dR(\xi) + \int_z^h e^{-(\xi^2/4\rho)} L_{n+p}^{\left(\upsilon-\frac{1}{2}\right)}\left(\frac{\xi^2}{4\rho}\right) \Psi(\xi,\rho)\,dR(\xi) + \int_h^\infty e^{-(\xi^2/4\rho)} L_{n+p}^{\left(\upsilon-\frac{1}{2}\right)}\left(\frac{\xi^2}{4\rho}\right) G(\xi,\rho)\,dR(\xi)\right] \qquad [36]$$

where , H(ξ,ρ) , G(ξ,ρ) and dR (ξ) are the same as defined by[30], [28] and [10].

The solution of quadruple equations involving generalized Languerre polynomials can be obtained independently by the above procedure [13-15].

4. CONCLUSION

By expanding the use of heat polynomials into the realm of quadruple series, this study opens new avenues for research and application, offering a deeper understanding of the mathematical underpinnings of heat conduction and its broader implications in various scientific fields. In this paper, we obtain the solution of the quadruple series equations involving generalized Laguerre polynomials as a special case of the equations considered in the present paper. The solution of quadruple equations involving generalized Languerre polynomials can be obtained independently by equations {32], [33], [34], [35] and [36].

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